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CHARACTERIZATION OF ε-NEAREST POINTS IN SPACES WITH ASYMMETRIC SEMINORM*

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Dedicated to professor Elena Popoviciu on the occasion of her 80th anniversary.

Abstract. In this note we are concerned with the characterization of the elements of ε-best approximation (ε-nearest points) in a subspace Y of space X with asymmetric seminorm. For this we use functionals in the asymmetric dual X^b defined and studied in some recent papers [1], [3], [5].

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1. INTRODUCTION

Let X be a real linear space. A functional $p : X \to [0, \infty)$ with the properties:

(1) $p(x) \geq 0$, for all $x \in X$,
(2) $p(tx) = tp(x)$, for all $x \in X$ and $t \geq 0$,
(3) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,

is called asymmetric seminorm on X, and the pair (X, p) is called a space with asymmetric seminorm.

The functional $\overline{p} : X \to [0, \infty)$, defined by $\overline{p}(x) = p(-x)$, $x \in X$ is another asymmetric seminorm on X, called the *conjugate of* p.

The functional $p^s : X \to [0, \infty)$, defined by

$$p^s(x) = \max\{p(x), p(-x)\}, x \in X,$$

is a seminorm on X. If p^s satisfies the axioms of a norm, then p is called an asymmetric norm on X. It follows that p satisfies the properties (1), (2), (3), and

(4) $p(x) = 0$ and $p(-x) = 0$ imply $x = 0$.

The asymmetric seminorm p on X generates a topology τ_p on X, having as a basis of neighborhoods of a point $x \in X$ the open p-balls

$$B'_p(x, r) = \{x' \in X : p(x' - x) < r\}, r > 0.$$

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The family of closed p-balls

$$B_p(x,r) = \{x' \in X : p(x'-x) \le r\}, r > 0,$$

generates the same topology. This topology τ_p could not be Hausdorff (see [5]), and could not be linear (the multiplication by scalars is not continuous in general, see [1]).

Let $\mathbb{R}$ be the set of real numbers and $u : \mathbb{R} \to [0,\infty)$, $u(a) = \max\{a, 0\}$, $a \in \mathbb{R}$. Then the function u is an asymmetric seminorm on $\mathbb{R}$ and, for $a \in \mathbb{R}$, the intervals $(-\infty, a+\varepsilon)$, $\varepsilon > 0$, form a basis of neighborhoods of $a \in \mathbb{R}$ in the topology τ_u. The conjugate asymmetric seminorm of u is $\overline{u} : \mathbb{R} \to [0,\infty)$, $\overline{u}(a) = u(-a)$, $a \in \mathbb{R}$, and $u^s(a) = \max\{u(a), u(-a)\} = |a|$ is a norm on $\mathbb{R}$. Consequently, u is an asymmetric norm on $\mathbb{R}$.

Let $\varphi : X \to \mathbb{R}$ be a linear functional. The continuity of φ with respect to the topologies τ_p and τ_u is called (p,u)-continuity, and it is equivalent to the upper semicontinuity of φ as a functional from (X, τ_p) to $(\mathbb{R}, |.|)$.

The linear functional $\varphi : (X, \tau_p) \to (\mathbb{R}, u)$ is (p,u)-continuous if and only if it is p-bounded, i.e. there exists $L \ge 0$ such that

$$\varphi(x) \le Lp(x), \quad \text{for all } x \in X.$$

The set of all (p,u)-continuous functionals is denoted by X_p^b. With respect to pointwise addition and multiplication by real scalars, the set X_p^b is a cone, i.e. $\lambda \ge 0$ and $\varphi, \psi \in X_p^b$ imply $\varphi + \psi \in X^b$ and $\lambda\varphi \in X_p^b$.

The functional $\|.| : X_p^b \to [0,\infty)$ defined by

$$\|\varphi|_p = \sup\{\varphi(x) : x \in X, p(x) \le 1\}, \quad \varphi \in X_p^b$$

satisfies the properties of an asymmetric seminorm, and the pair $(X_p^b, \|.|_p)$ is called the asymmetric dual of the asymmetric seminormed space (X,p) (see [5]). Some properties of this dual are presented in [1], [3], [5]. If there is no danger of confusion we shall use the notation X^b and $\|\varphi|$ instead of X_p^b and $\|\varphi|_p$, respectively.

Let (X,p) be an asymmetric seminormed space and Y a subspace of X. Let Y^b be the asymmetric dual of (Y,p).

The following result is the analog of a well known extension result for linear functionals in normed spaces.

Theorem 1 (Hahn-Banach). *Let (Y,p) be a subspace of asymmetric seminormed space (X,p). Then for every $\varphi_0 \in Y^b$ there exists $\varphi \in X^b$ such that*

$$\varphi|_Y = \varphi_0,$$
$$\|\varphi| = \|\varphi_0|.$$

Proof. We consider the functional $q : X \to [0,\infty)$, $q(x) = \|\varphi_0| \cdot p(x)$, $x \in X$. Obviously q is subadditive and positive homogeneous and for every $y \in Y$ we have

$$\varphi_0(y) \le \|\varphi_0| \cdot p(y) = q(y)$$

i.e. φ_0 is majorized by q on Y.

By Hahn-Banach extension theorem it results that there exists the linear functional $\varphi : X \to \mathbb{R}$ with properties:

$$\varphi|_Y = \varphi_0 \quad \text{and}$$
$$\varphi(x) \leq \|\varphi_0| \cdot p(x), \text{ for every } x \in X.$$

It follows $\|\varphi| \leq \|\varphi_0|$, and, because

$$\begin{aligned} \|\varphi| &= \sup\{\varphi(x) : x \in X, p(x) \leq 1\} \\ &\geq \sup\{\varphi(y) : y \in Y,\ p(y) \leq 1\} \\ &= \sup\{\varphi_0(y) : y \in Y,\ p(y) \leq 1\} \\ &= \|\varphi_0| \end{aligned}$$

we have $\|\varphi| = \|\varphi_0|$. □

2. THE ε-BEST APPROXIMATION IN (X, p)

Let Y be a nonvoid subset of the asymmetric seminormed space (X, p).

The problem of best approximation of the element $x \in X$ by elements in Y is: find an element $y_0 \in Y$ such that

(1) $$d_p(x, Y) := \inf\{p(y - x) : y \in Y\} = p(y_0 - x).$$

Let $\varepsilon > 0$. The problem of ε-best approximation of $x \in X$ by elements in Y is: find $y_0 \in Y$ such that

(2) $$p(y_0 - x) \leq d_p(x, Y) + \varepsilon.$$

Obviously, the problem of ε-best approximation always admits a solution, because for every number $n \in \mathbb{N}$ there exists $y_n \in Y$ such that $p(y_n - x) \leq d_p(x, Y) + \frac{1}{n}$, so that $p(y_n - x) \leq d_p(x, Y) + \varepsilon$, for $n > \left[\frac{1}{\varepsilon}\right] + 1$.

In the following we denote by

(3) $$P_{Y,\varepsilon}(x) = \{y \in Y : p(y - x) \leq d_p(x, Y) + \varepsilon\},\ x \in X$$

the nonvoid set of the elements of ε-best approximation for $x \in X$ in Y.

The paper [3] contains characterizations, in terms of functionals in X^b vanishing on Y, of the elements of best approximation of $x \in X$ by elements in a subspace Y of X. Let us observe firstly that, one can consider also the problem of ε-best approximation by using the conjugate $\overline{p}$ of p. In this case, for

(4) $$d_{\overline{p}}(x, Y) = \inf\{p(x - y) : y \in Y\}$$

one looks for $y_0 \in Y$ such that

(5) $$p(x - y_0) \leq d_{\overline{p}}(x, Y) + \varepsilon.$$

Let us denote by

(6) $$\overline{P}_{Y,\varepsilon}(x) = \{y \in Y : \overline{p}(y - x) = p(x - y) \leq d_{\overline{p}}(x, Y) + \varepsilon\}$$

the set of ε-best approximation of $x \in X$ with respect to the conjugate asymmetric seminorm $\overline{p}$.

In the following we obtain characterizations of elements of ε-best approximation of $x \in X$ by elements of a subspace Y, with respect to the asymmetric seminorms p and $\overline{p}$.

Results of this type, for elements of best approximations in a normed space X, using the elements of dual X^* are obtained in [17] (see also [3], [9], [11], [13], [16], [18]).

Concerning the characterizations of elements of ε-best approximation in normed space, see papers [14], [15].

THEOREM 2. *Let (X,p) be an asymmetric seminormed space, Y a subspace of X and $x_0 \in X \setminus Y$, such that $d = d_p(x_0, Y) > 0$ and $\overline{d} = d_{\overline{p}}(x_0, Y) > 0$. Then*

(a) *An element $y_0 \in Y$ is in $P_{Y,\varepsilon}(x_0)$ if and only if there exists $\varphi \in X_p^b$ with the properties:*
  (i) $\varphi(y) = 0$, *for all* $y \in Y$,
  (iii) $\|\varphi|_p = 1$,
  (iii) $\varphi(-x_0) \geq p(y_0 - x_0) - \varepsilon$.

(b) *An element y_0 is in $\overline{P}_{Y,\varepsilon}(x_0)$ if and only if there exists $\psi \in X_p^b$ with the properties:*
  (j) $\psi(y) = 0$, *for all* $y \in Y$,
  (jj) $\|\psi|_p = 1$,
  (jjj) $\psi(x_0) \geq p(x_0 - y_0) - \varepsilon$.

Proof. Let $x_0 \in X \setminus Y$ and $Z = Y + \langle x_0 \rangle$ be the direct sum of Y with the space generated by x_0. Consider the functional $\varphi_0 : Z \to \mathbb{R}$ defined by

$$\varphi_0(z) = \varphi(y + \lambda x_0) = -\lambda,$$

where $z \in Z$, and z is uniquely represented in the form $z = y + \lambda x_0$.

The functional φ_0 is linear on Z.

Observe that $\varphi_0 \mid_Y = 0$, and for every $\lambda > 0$ we have

$$p(y - \lambda x_0) = \lambda p\left(\tfrac{1}{\lambda} y - x_0\right) \geq \lambda d = d \cdot \varphi_0(y - \lambda x_0).$$

It follows that

$$\varphi_0(y - \lambda x_0) \leq \tfrac{1}{d} \cdot p(y - \lambda x_0),$$

for every $\lambda > 0$.

Because the last inequality is also valid if $\varphi_0(y - t x_0) = t \leq 0$, it follows

$$\|\varphi_0|_p \leq \tfrac{1}{d}, \text{ and consequently } \varphi_0 \in Z_p^b.$$

Now, let $(y_n)_{n \geq 1}$ be a sequence in Y such that $p(y_n - x_0) \to d$, for $n \to \infty$, and such that $p(y_n - x_0) > 0$ for every $n \in \mathbb{N}$. Then

$$\|\varphi_0|_p \geq \varphi_0\left(\tfrac{y_n - x_0}{p(y_n - x_0)}\right) = \tfrac{1}{p(y_n - x_0)} \to \tfrac{1}{d},$$

and, consequently, $\|\varphi_0|_p = \frac{1}{d}$.

By Theorem 1, there exists $\varphi_1 \in X^b$ such that

$$\varphi_1 \mid_Z = \varphi_0, \quad \|\varphi_1| = \|\varphi_0|_p = \frac{1}{d}.$$

Then, the functional $\varphi = d \cdot \varphi_1$ satisfies the properties: $\varphi \in X_p^b$, $\varphi \mid_Y = d \cdot \varphi_1|_Y = 0$,

$$\begin{aligned} \varphi(-x_0) &= \varphi(y_0) + \varphi(-x_0) \\ &= \varphi(y_0 - x_0) \\ &\geq p(y_0 - x_0) \\ &\geq p(y_0 - x_0) - \varepsilon. \end{aligned}$$

Conversely, if $y_0 \in Y$ and there exists $\varphi \in X_p^b$ with the properties (a) (i)-(iii), then for every $y \in Y$ we have

$$\begin{aligned} p(y_0 - x_0) &\leq \varphi(-x_0) + \varepsilon \\ &= \varphi(y - x_0) + \varepsilon \\ &\leq \|\varphi|_p \cdot p(y - x_0) + \varepsilon \\ &\leq p(y - x_0) + \varepsilon. \end{aligned}$$

Taking the infimum with respect to $y \in Y$, one finds

$$p(y_0 - x_0) \leq d_p(x_0, Y) + \varepsilon;$$

so that $y_0 \in P_{Y,\varepsilon}(x_0)$.

Similarly, defining $\psi_0 : Z = Y + \langle x_0 \rangle \to \mathbb{R}$ by $\psi(z) = \psi_0(y + \lambda x_0) = \lambda, \quad y \in Y$ and $\lambda \in \mathbb{R}$, and proceeding in the same way, one obtains the claim (b) of the theorem. □

Theorem 2 has the following consequence:

COROLLARY 3. *In the hypothesis of Theorem 2 we have:*

(a′) $M \subset P_{Y,\varepsilon}(x_0)$, *if and only if there exists* $\varphi \in X^b$ *verifying (a) (i)-(ii) and the condition:*

$$\varphi(-x_0) \geq p(u - x_0) - \varepsilon, \quad \textit{for all} \quad u \in M;$$

(b′) $M \subset P_{Y,\varepsilon}(x_0)$ *if and only if there exists* $\psi \in X^b$ *with properties (b) (j)-(jj), and verifying the condition:*

$$\psi(x_0) \geq p(x_0 - u) - \varepsilon, \quad \textit{for all} \quad u \in M.$$

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