ITERATED FUNCTION SYSTEM OF LOCALLY CONTRACTIVE OPERATORS

ADRIAN PETRUSEL

Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. The aim of this paper is to study the properties of the fractal and the multi-fractal operator generated by some iterated function system satisfying to a locally contractive type condition.


Keywords. Fixed point, self-similar set, locally contractive type operator.

1. BASIC NOTIONS AND RESULTS

For the convenience of the reader, some notations and basic notions are first presented.

Let $(X,d)$ be a metric space. We consider the following spaces of subsets of a metric space $(X,d)$:

- $\mathcal{P}(X) = \{Y | Y \subset X\}$,
- $P(X) = \{Y \in \mathcal{P}(X) | Y \neq \emptyset\}$,
- $P_d(X) = \{Y \in P(X) | Y \text{ closed}\}$,
- $P_{cp}(X) = \{Y \in P(X) | Y \text{ compact}\}$.

Let us consider now some (generalized) functionals on $\mathcal{P}(X)$:

1. The gap functional $D : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$
   
   $$D(A,B) = \begin{cases} \inf \{d(a,b) | a \in A, b \in B\}, & \text{if } A \neq \emptyset \neq B, \\ 0, & \text{if } A = \emptyset = B, \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

2. The excess functional $\rho : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$
   
   $$\rho(A,B) = \begin{cases} \sup \{D(a,B) | a \in A\}, & \text{if } A \neq \emptyset \neq B, \\ 0, & \text{if } A = \emptyset, \\ +\infty, & \text{if } B = \emptyset \neq A. \end{cases}$$

*“Babeș-Bolyai” University Cluj-Napoca, Department of Applied Mathematics, Kogălniceanu 1, 400084 Cluj-Napoca, Romania, e-mail: petrusel@math.ubbcluj.ro.
(3) Pompeiu-Hausdorff generalized functional \( H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\} \),

\[
H(A, B) = \begin{cases} \max\{\rho(A, B), \rho(B, A)\}, & \text{if } A \neq \emptyset \neq B, \\ 0, & \text{if } A = \emptyset = B, \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}
\]

It is known the fact that \( H \) is a generalized metric on the space of all nonempty closed subsets of a metric space and the space \( (P_2(X), H) \) is complete provided that the metric space \( (X, d) \) is complete.

A metric space \( (X, d) \) is said to be \( \varepsilon \)-chainable (where \( \varepsilon > 0 \) is fixed) if and only if, given \( a, b \in X \), there is an \( \varepsilon \)-chain from \( a \) to \( b \), that is a finite set of points \( x_0, x_1, \ldots, x_n \) in \( X \) such that \( x_0 = a \), \( x_n = b \) and \( d(x_{i-1}, x_i) < \varepsilon \), for all \( i \in \{1, 2, \ldots, n\} \).

If \( f : X \to X \) is a single-valued operator, then \( x^* \in X \) is a fixed point for \( f \) if \( x^* = f(x^*) \). We will denote by \( \text{Fix} f \) the fixed points set of \( f \).

If \( F : X \to \mathcal{P}(X) \) is a multi-valued operator then \( x^* \in X \) is a fixed point for \( F \) if \( x^* \in F(x^*) \). We will denote by \( \text{Fix} F \) the fixed points set of \( F \).

The following notion is important for our main results.

**Definition 1.** Let \( \varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) be two mappings. Then \( \psi \) is said to be strong \( \varphi \)-summable if:

i) \( \varphi \) is monotone increasing

ii) \( \psi \circ \varphi \) is monotone increasing

iii) for each \( t \in \mathbb{R}_+ \) the sequence \( (\varphi^n(t))_{n \in \mathbb{N}} \) converges to 0, as \( n \to \infty \) and \( \sum_{n=1}^{\infty}(\psi \circ \varphi)^n(t) < \infty \)

iv) \( \psi \) is an expansion function, i.e. \( \psi(0) = 0 \) and \( \psi(t) > t \), for each \( t > 0 \).

**Example.** Let \( \varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+ \), defined by \( \varphi(t) = at \) (where \( a \in [0, 1] \)) and \( \psi(t) = bt \) (with \( b \in [1, \frac{1}{a}] \)), for each \( t \in \mathbb{R}_+ \). Then \( \psi \) is said to be strong \( \varphi \)-summable.

If \( f_i, i \in \{1, \ldots, m\} \) are continuous operators of \( X \) into itself, then a nonempty compact set \( Y \) in \( X \) is said to be self-similar if it satisfies the condition \( Y = \bigcup_{i=1}^{m} f_i(Y) \). The above relation can be considered also as a fixed point problem for a suitable operator.

**Definition 2.** Let \( f_i : X \to X, i \in \{1, \ldots, m\} \) be a finite family of continuous operators. Let us define

\[
T_f : (P_{cp}(X), H) \to (P_{cp}(X), H), \quad T_f(Y) = \bigcup_{i=1}^{m} f_i(Y).
\]

Then, \( T_f \) is the fractal operator generated by the iterated function system \( f = (f_1, f_2, \ldots, f_m) \).

The Hausdorff dimension of a self-similar set \( Y \) is not, in general, an integer. For this reason, \( Y \) is a fractal and \( P_{cp}(X) \) is called the space of fractals.

Let consider now the case of multi-valued operators.
Definition 3. Let $F_1, \ldots, F_m : X \to P_{cp}(X)$ be a finite family of upper semi-continuous (briefly u.s.c.) multi-valued operators. We define the multi-fractal operator $T_F$ generated by the iterated multi-functions system $F = (F_1, F_2, \ldots, F_m)$, by the following relation:

$$T_F : P_{cp}(X) \to P_{cp}(X), \quad T_F(Y) = \bigcup_{i=1}^{m} F_i(Y).$$

A nonempty compact subset $A^*$ of $X$ is said to be a multi-self-similar set for the iterated multi-functions system $F = (F_1, F_2, \ldots, F_m)$ if and only if it is a fixed point for the associated multi-fractal operator.

If $F = (F_1, F_2, \ldots, F_m)$ is a finite family of continuous single-valued operators then a fixed point of the corresponding fractal operator $T_F$ will be called a self-similar set.

We consider now some contractive type conditions for a single-valued operator $f : X \to X$.

Definition 4. The single-valued operator $f : X \to X$ satisfies

i) $\epsilon$-locally contractive condition (where $\epsilon > 0$) if there is $\alpha \in [0, 1[$ such that for $x, y \in X$, $d(x, y) < \epsilon$ we have $d(f(x), f(y)) \leq \alpha d(x, y)$

ii) $\epsilon$-locally Meir-Keeler type condition (where $\epsilon > 0$) if for each $0 < \eta < \epsilon$ there is $\delta > 0$ such that for $x, y \in X$, $\eta \leq d(x, y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$.

iii) $\epsilon$-locally Boyd-Wong type condition (where $\epsilon > 0$) if for each $x, y \in X$ with $0 < d(x, y) < \epsilon$ we have $d(f(x), f(y)) \leq k d(x, y)$, where $k : [0, \infty) \to [0, 1[$ is a upper semi-continuous function with the property $k(t) < t$, for each $t \in ]0, \epsilon[$.

iv) $(\epsilon, \varphi)$-locally contractive condition (where $\epsilon > 0$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$) if for each $x, y \in X$ and $0 < \alpha \leq \epsilon$ with $d(x, y) < \alpha$ implies $d(f(x), f(y)) \leq \varphi(\alpha)$.

Let us observe that (i) implies (ii). Indeed, for each $0 < \eta < \epsilon$ we can choose $\delta := \min\{\eta/2, \epsilon - \eta\}$. Then, if $x, y \in X$ with $\eta \leq d(x, y) < \eta + \delta$ we obtain $d(f(x), f(y)) \leq k d(x, y) < k \eta < \eta$.

Also (iii) implies (ii), while (ii) implies (iv). For other contractive type conditions and the relations between them we refer to [5] (see also [1],[6],[8],[9],[11]).

Some contractive type conditions for multi-valued operators on a metric space $(X, d)$ are contained in the following definition.

Definition 5. The multi-valued operator $F : X \to P_{cl}(X)$ is said to be:

i) multi-valued $\epsilon$-locally contractive condition (where $\epsilon > 0$) if there is $\alpha \in [0, 1[$ such that for $x, y \in X$, $d(x, y) < \epsilon$ we have $H(F(x), F(y)) \leq \alpha d(x, y)$

ii) multi-valued $\epsilon$-locally Meir-Keeler type operator (where $\epsilon > 0$) if for each $0 < \eta < \epsilon$ there is $\delta > 0$ such that for $x, y \in X$, $\eta \leq d(x, y) < \eta + \delta$ we have $H(F(x), F(y)) < \eta$. 

(iii) multi-valued \(\epsilon\)-locally Boyd-Wong type operator \(\) (where \(\epsilon > 0\)) if for each \(x, y \in X\) with \(0 < d(x, y) < \epsilon\) we have \(H(F(x), F(y)) \leq k(d(x, y))\), where \(k : [0, \infty) \to [0, 1]\) is an upper semi-continuous function with the property \(k(t) < t\), for each \(t \in [0, \epsilon]\).

(iv) multi-valued \((\epsilon, \varphi)\)-locally contractive operator \(\) (where \(\epsilon > 0\) and \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\)) if \(x, y \in X\) and \(0 < \alpha \leq \epsilon\) with \(d(x, y) < \alpha\) implies \(H(F(x), F(y)) \leq \varphi(\alpha)\).

We have to remark that (i) implies (ii), (iii) implies (ii), while (ii) implies (iv). For other conditions of this type and several results see \[2\], \[3\], \[4\], \[9\], \[10\], \[12\], \[13\].

2. SELF-SIMILAR AND MULTI-SELF-SIMILAR SETS

We start this section by recalling the following fixed point result:

**THEOREM 6** (Petruşel \[7\]). Let \((X, d)\) be an \(\epsilon\)-chainable complete metric space \((\epsilon > 0)\), \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) a strong \(\varphi\)-summable function and \(f : X \to X\) a single-valued operator satisfying to a \((\epsilon, \varphi)\)-locally contractive condition. Then \(\text{Fix} f \neq \emptyset\).

The existence result for a self-similar set of an iterated function system satisfying to a locally contractive type condition is:

**THEOREM 7.** Let \((X, d)\) be an \(\epsilon\)-chainable complete metric space \((\epsilon > 0),\) \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) a strong \(\varphi\)-summable function and \(F_1, F_2, \ldots, F_m : X \to X\) be a finite family of single-valued operators satisfying to an \((\epsilon, \varphi)\)-locally contractive condition. Then the fractal operator \(T_f\) is an \((\epsilon, \varphi)\)-locally contractive type operator; having at least a fixed point.

**Proof.** We will prove that for each \(A, B \in P_{cp}(X)\) and \(0 < \alpha \leq \epsilon\) with \(H(A, B) < \alpha\) we have \(H(T_f(A), T_f(B)) \leq \varphi(\alpha)\). For this purpose let \(A, B \in P_{cp}(X)\) such that \(0 < \alpha \leq \epsilon\) with \(H(A, B) < \alpha\). We intend to prove that for each \(u \in T_f(A)\) there is \(v \in T_f(B)\) such that \(d(u, v) \leq \varphi(\alpha)\). For \(u \in T_f(A)\) there exists \(j \in \{1, 2, \ldots, m\}\) such that \(u \in f_j(A)\). Then we can find \(a \in A\) such that \(u = f_j(a)\). Since \(A, B\) are compact sets, for \(a \in A\) there exists \(b \in B\) such that \(d(a, b) \leq H(A, B) < \alpha\). Hence \(d(f_j(a), f_j(b)) \leq \varphi(\alpha)\). So, if we define \(v := f_j(b)\) we got that \(d(u, v) \leq \varphi(\alpha)\). By interchanging the roles of \(u\) and \(v\) we obtain the desired conclusion.

The final conclusion follows now from Theorem 6.

An existence result for a self-similar set of an iterated multi-function system satisfying to a locally contractive type condition is:

**THEOREM 8.** Let \((X, d)\) be an \(\epsilon\)-chainable complete metric space \((\epsilon > 0),\) \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) a strong \(\varphi\)-summable function and \(F_1, F_2, \ldots, F_m : X \to P_{cp}(X)\) be a finite family of multi-valued operators satisfying to a multi-valued \((\epsilon, \varphi)\)-locally contractive condition. Then the fractal operator \(T_F\) is an \((\epsilon, \varphi)\)-locally contractive type operator; having at least a fixed point.
Proof. There are only minor modifications of the previous proof. More precisely, we will show that for $A, B \in P_{cp}(X)$ and $0 < \alpha \leq \epsilon$ with $H(A, B) < \alpha$ we have that $H(T_F(A), T_F(B)) \leq \varphi(\alpha)$. In this respect, let $A, B \in P_{cp}(X)$ such that $0 < \alpha \leq \epsilon$ with $H(A, B) < \alpha$. We intend to prove that for each $u \in T_F(A)$ there is $v \in T_F(B)$ such that $d(u, v) \leq \varphi(\alpha)$. For $u \in T_F(A)$ there exists $j \in \{1, 2, \ldots, m\}$ such that $u \in F_j(a)$. Then we can find $a \in A$ such that $d(a, b) \leq H(A, B) < \alpha$. Hence $H(F_j(a), F_j(b)) \leq \varphi(\alpha)$. So, we can choose $v \in F_j(b)$ such that $d(u, v) \leq \varphi(\alpha)$. By interchanging the roles of $u$ and $v$ we obtain that $H(T_F(A), T_F(B)) \leq \varphi(\alpha)$.

The conclusion is again an immediate application of Theorem 6.

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REFERENCES


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