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ITERATED FUNCTION SYSTEM OF LOCALLY CONTRACTIVE OPERATORS

ADRIAN PETRUŞEL*

Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. The aim of this paper is to study the properties of the fractal and the multi-fractal operator generated by some iterated function system satisfying to a locally contractive type condition.

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1. BASIC NOTIONS AND RESULTS

For the convenience of the reader, some notations and basic notions are first presented.

Let (X,d) be a metric space. We consider the following spaces of subsets of a metric space (X,d):

$$\begin{aligned}\mathcal{P}(X) &= \{Y | Y \subset X\},\\ P(X) &= \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\},\\ P_{cl}(X) &= \{Y \in P(X) |\ Y \text{closed}\},\\ P_{cp}(X) &= \{Y \in P(X) |\ Y \text{compact}\}.\end{aligned}$$

Let us consider now some (generalized) functionals on $\mathcal{P}(X)$:

(1) The gap functional $D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$D(A,B) = \begin{cases} \inf\{d(a,b) |\ a \in A,\ b \in B\}, & \text{if } A \neq \emptyset \neq B,\\ 0, & \text{if } A = \emptyset = B,\\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B.\end{cases}$$

(2) The excess functional $\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\rho(A,B) = \begin{cases} \sup\{D(a,B) |\ a \in A\}, & \text{if } A \neq \emptyset \neq B,\\ 0, & \text{if } A = \emptyset,\\ +\infty, & \text{if } B = \emptyset \neq A\end{cases}$$

*"Babeş-Bolyai" University Cluj-Napoca, Department of Applied Mathematics, Kogălniceanu 1, 400084 Cluj-Napoca, Romania, e-mail: petrusel@math.ubbcluj.ro.

(3) Pompeiu-Hausdorff generalized functional $H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H(A,B) = \begin{cases} \max\{\rho(A,B), \rho(B,A)\}, & \text{if } A \neq \emptyset \neq B, \\ 0, & \text{if } A = \emptyset = B, \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

It is known the fact that H is a generalized metric on the space of all nonempty closed subsets of a metric space and the space $(P_{cl}(X), H)$ is complete provided that the metric space (X, d) is complete.

A metric space (X, d) is said to be ε-chainable (where $\varepsilon > 0$ is fixed) if and only if, given $a, b \in X$, there is an ε-chain from a to b, that is a finite set of points $x_0, x_1, \ldots, x_n$ in X such that $x_0 = a$, $x_n = b$ and $d(x_{i-1}, x_i) < \varepsilon$, for all $i \in \{1, 2, \ldots, n\}$.

If $f : X \to X$ is a single-valued operator, then $x^* \in X$ is a fixed point for f if $x^* = f(x^*)$. We will denote by $Fixf$ the fixed points set of f.

If $F : X \to P(X)$ is a multi-valued operator then $x^* \in X$ is a fixed point for F if $x^* \in F(x^*)$. We will denote by $FixF$ the fixed points set of F.

The following notion is important for our main results.

Definition 1. *Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be two mappings. Then ψ is said to be strong φ-summable if:*

i) *φ is monotone increasing*

ii) *$\psi \circ \varphi$ is monotone increasing*

iii) *for each $t \in \mathbb{R}_+$ the sequence $(\varphi^n(t))_{n \in \mathbb{N}}$ converges to 0, as $n \to \infty$ and $\sum_{n \geq 1} (\psi \circ \varphi)^n(t) < \infty$*

iv) *ψ is an expansion function, i.e. $\psi(0) = 0$ and $\psi(t) > t$, for each $t > 0$.*

Example. Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$, defined by $\varphi(t) = at$ (where $a \in [0, 1[$) and $\psi(t) = bt$ (with $b \in]1, \frac{1}{a}[$), for each $t \in \mathbb{R}_+$. Then ψ is said to be strong φ-summable. □

If f_i, $i \in \{1, \ldots, m\}$ are continuous operators of X into itself, then a nonempty compact set Y in X is said to be **self-similar** if it satisfies the condition $Y = \bigcup_{i=1}^{m} f_i(Y)$. The above relation can be considered also as a fixed point problem for a suitable operator.

Definition 2. *Let $f_i : X \to X$, $i \in \{1, \ldots, m\}$ be a finite family of continuous operators. Let us define*

$$T_f : (P_{cp}(X), H) \to (P_{cp}(X), H), \qquad T_f(Y) = \bigcup_{i=1}^{m} f_i(Y).$$

Then, T_f is the fractal operator generated by the iterated function system $f = (f_1, f_2, \ldots, f_m)$.

The Hausdorff dimension of a self-similar set Y is not, in general, an integer. For this reason, Y is a fractal and $P_{cp}(X)$ is called the space of fractals.

Let consider now the case of multi-valued operators.

Definition 3. *Let $F_1, \dots, F_m : X \to P_{cp}(X)$ be a finite family of upper semi-continuous (briefly u.s.c.) multi-valued operators. We define the multi-fractal operator T_F generated by the iterated multi-functions system $F = (F_1, F_2, \dots, F_m)$, by the following relation:*

$$T_F : P_{cp}(X) \to P_{cp}(X), \quad T_F(Y) = \bigcup_{i=1}^{m} F_i(Y).$$

A nonempty compact subset A^* of X is said to be a **multi-self-similar set** *for the iterated multi-functions system $F = (F_1, F_2, \dots, F_m)$ if and only if it is a fixed point for the associated multi-fractal operator.*

If $F = (F_1, F_2, \dots, F_m)$ is a finite family of continuous single-valued operators then a fixed point of the corresponding fractal operator T_F will be called a self-similar set.

We consider now some contractive type conditions for a single-valued operator $f : X \to X$.

Definition 4. *The single-valued operator $f : X \to X$ satisfies*

i) **ϵ-locally contractive condition** *(where $\epsilon > 0$) if there is $\alpha \in [0, 1[$ such that for $x, y \in X$, $d(x, y) < \epsilon$ we have $d(f(x), f(y)) \le \alpha d(x, y)$*

ii) **ϵ-locally Meir-Keeler type condition** *(where $\epsilon > 0$) if for each $0 < \eta < \epsilon$ there is $\delta > 0$ such that $x, y \in X$, $\eta \le d(x, y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$.*

iii) **ϵ-locally Boyd-Wong type condition** *(where $\epsilon > 0$) if for each $x, y \in X$ with $0 < d(x, y) < \epsilon$ we have $d(f(x), f(y)) \le k(d(x, y))$, where $k : [0, \infty) \to [0, 1[$ is a upper semi-continuous function with the property $k(t) < t$, for each $t \in]0, \epsilon[$.*

iv) **(ϵ, φ)-locally contractive condition** *(where $\epsilon > 0$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$) if for each $x, y \in X$ and $0 < \alpha \le \epsilon$ with $d(x, y) < \alpha$ implies $d(f(x), f(y)) \le \varphi(\alpha)$.*

Let us observe that (i) implies (ii). Indeed, for each $0 < \eta < \epsilon$ we can choose $\delta := \min\{\frac{\eta(1-k)}{k}, \epsilon - \eta\}$. Then, if $x, y \in X$ with $\eta \le d(x, y) < \eta + \delta$ we obtain $d(f(x), f(y)) \le kd(x, y) < k\frac{\eta}{k} < \eta$.

Also (iii) implies (ii), while (ii) implies (iv). For other contractive type conditions and the relations between them we refer to [5] (see also [1],[6],[8],[9],[11]).

Some contractive type conditions for multi-valued operators on a metric space (X, d) are contained in the following definition.

Definition 5. *The multi-valued operator $F : X \to P_{cl}(X)$ is said to be:*

i) **multi-valued ϵ-locally contractive condition** (*where $\epsilon > 0$*) *if there is $\alpha \in [0, 1[$ such that for $x, y \in X$, $d(x, y) < \epsilon$ we have $H(F(x), F(y)) \le \alpha d(x, y)$*

(ii) **multi-valued ϵ-locally Meir-Keeler type operator** *(where $\epsilon > 0$) if for each $0 < \eta < \epsilon$ there is $\delta > 0$ such that $x, y \in X$, $\eta \le d(x, y) < \eta + \delta$ we have $H(F(x), F(y)) < \eta$.*

(iii) **multi-valued ϵ-locally Boyd-Wong type operator** *(where $\epsilon > 0$) if for each $x, y \in X$ with $0 < d(x, y) < \epsilon$ we have $H(F(x), F(y)) \leq k(d(x, y))$, where $k : [0, \infty) \to [0, 1[$ is an upper semi-continuous function with the property $k(t) < t$, for each $t \in]0, \epsilon[$.*

(iv) **multi-valued (ϵ, φ)-locally contractive operator** *(where $\epsilon > 0$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$) if $x, y \in X$ and $0 < \alpha \leq \epsilon$ with $d(x, y) < \alpha$ implies $H(F(x), F(y)) \leq \varphi(\alpha)$.*

We have to remark that (i) implies (ii), (iii) implies (ii), while (ii) implies (iv). For other conditions of this type and several results see [2], [3], [4], [9],[10], [12], [13].

2. SELF-SIMILAR AND MULTI-SELF-SIMILAR SETS

We start this section by recalling the following fixed point result:

THEOREM 6 (Petruşel [7]). *Let (X, d) be an ϵ-chainable complete metric space (where $\epsilon > 0$), $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a strong φ-summable function and $f : X \to X$ a single-valued operator satisfying to a (ϵ, φ)-locally contractive condition. Then $Fixf \neq \emptyset$.*

The existence result for a self-similar set of a iterated function system satisfying to a locally contractive type condition is:

THEOREM 7. *Let (X, d) be an ϵ-chainable complete metric space (where $\epsilon > 0$), $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a strong φ-summable function and $F_1, F_2, \ldots, F_m : X \to X$ be a finite family of single-valued operators satisfying to an (ϵ, φ)-locally contractive condition. Then the fractal operator T_f is an (ϵ, φ)-locally contractive type operator, having at least a fixed point.*

Proof. We will prove that for each $A, B \in P_{cp}(X)$ and $0 < \alpha \leq \epsilon$ with $H(A, B) < \alpha$ we have $H(T_f(A), T_f(B)) \leq \varphi(\alpha)$. For this purpose let $A, B \in P_{cp}(X)$ such that $0 < \alpha \leq \epsilon$ with $H(A, B) < \alpha$. We intend to prove that for each $u \in T_f(A)$ there is $v \in T_f(B)$ such that $d(u, v) \leq \varphi(\alpha)$. For $u \in T_f(A)$ there exists $j \in \{1, 2, ..., m\}$ such that $u \in f_j(A)$. Then we can find $a \in A$ such that $u = f_j(a)$. Since A, B are compact sets, for $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B) < \alpha$. Hence $d(f_j(a), f_j(b)) \leq \varphi(\alpha)$. So, if we define $v := f_j(b)$ we got that $d(u, v) \leq \varphi(\alpha)$. By interchanging the roles of u and v we obtain the desired conclusion.

The final conclusion follows now from Theorem 6. □

An existence result for a self-similar set of a iterated multi-function system satisfying to a locally contractive type condition is:

THEOREM 8. *Let (X, d) be an ϵ-chainable complete metric space (where $\epsilon > 0$), $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a strong φ-summable function and $F_1, F_2, ..., F_m : X \to P_{cp}(X)$ be a finite family of multi-valued operators satisfying to a multi-valued (ϵ, φ)-locally contractive condition. Then the fractal operator T_F is an (ϵ, φ)-locally contractive type operator, having at least a fixed point.*

Proof. There are only minor modifications of the previous proof. More precisely, we will show that for $A, B \in P_{cp}(X)$ and $0 < \alpha \leq \epsilon$ with $H(A, B) < \alpha$ we have that $H(T_F(A), T_F(B)) \leq \varphi(\alpha)$. In this respect, let $A, B \in P_{cp}(X)$ such that $0 < \alpha \leq \epsilon$ with $H(A, B) < \alpha$. We intend to prove that for each $u \in T_F(A)$ there is $v \in T_F(B)$ such that $d(u, v) \leq \varphi(\alpha)$. For $u \in T_F(A)$ there exists $j \in \{1, 2, ..., m\}$ such that $u \in F_j(A)$. Then we can find $a \in A$ such that $u \in F_j(a)$. Since A, B are compact sets, for $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B) < \alpha$. Hence $H(F_j(a), F_j(b)) \leq \varphi(\alpha)$. So, we can choose $v \in F_j(b)$ such that $d(u, v) \leq \varphi(\alpha)$. By interchanging the roles of u and v we obtain that $H(T_F(A), T_F(B)) \leq \varphi(\alpha)$.

The conclusion is again an immediate application of Theorem 6. □

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