ON A MEAN VALUE THEOREM CONNECTED WITH HERMITE-HADAMARD’S INEQUALITY

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Abstract. In this paper we prove a mean-value theorem for integral calculus, then we demonstrate properties of the mean point. In the end we give an extension of Hermite-Hadamard’s inequality.


Keywords. Convex function, mean-value theorem, intermediate point.

1. INTRODUCTION

In this article, we will give two interpolations of the Hermite-Hadamard’s inequality. We start from:

THEOREM 1 (Hermite-Hadamard). Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function. Then

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

For the proofs we refer to [3] or [8].

2. PRELIMINARIES

It is known that if $a < b$, then $x \in [a, b]$ iff there exists $t \in [0, 1]$ such that $x = (1-t)a + tb$, and if $x_1, x_2 \in [a, b]$, $x_1 \leq x_2$ are symmetrical with respect to the middle point of the interval $[a, b]$ iff there exists $c \in \left[0, \frac{b-a}{2}\right]$ such that $x_1 = \frac{a+b}{2} - c$ and $x_2 = \frac{a+b}{2} + c$.

Lemma 2. Let $[a, b]$, $a < b$, be an interval. The points $x_1, x_2 \in [a, b]$, $x_1 \leq x_2$ are symmetrical with respect to the middle point of the interval $[a, b]$ iff there exists $t \in \left[0, \frac{1}{2}\right]$, such that $x_1 = (1-t)a + tb$, and $x_2 = (1-t)b + ta$.

Proof. Let $t = \frac{b-a-c}{b-a} = \frac{1}{2} - \frac{c}{b-a}$. It can be checked that $t \in \left[0, \frac{1}{2}\right]$, and that $x_1 = (1-t)a + tb$, $x_2 = (1-t)b + ta$. Reciprocally, it is proved that $x_1 + x_2 = \frac{a+b}{2}$. \hfill \square

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3. MAIN RESULTS

THEOREM 3. Let \( f : [a, b] \to \mathbb{R} \) be a convex function. Then the function
\[
g : \left[0, \frac{b-a}{2}\right] \to \mathbb{R}, \quad g(x) = f \left( \frac{a+b}{2} - x \right) + f \left( \frac{a+b}{2} + x \right), \quad \forall x \in \left[0, \frac{b-a}{2}\right]
\]
is increasing on \( \left[0, \frac{b-a}{2}\right] \).

Proof. Let \( 0 \leq x_1 < x_2 \leq \frac{b-a}{2} \). Then
\[
a \leq \frac{a+b}{2} - x_2 < \frac{a+b}{2} - x_1 \leq \frac{a+b}{2} + x_1 < \frac{a+b}{2} + x_2 \leq b,
\]
and considering Lemma 2, we find that there exists \( t \in (0, \frac{1}{2}) \) so that \( \frac{a+b}{2} - x_1 = (1-t) \left( \frac{a+b}{2} - x_2 \right) + t \left( \frac{a+b}{2} + x_2 \right) \), and \( \frac{a+b}{2} + x_1 = (1-t) \left( \frac{a+b}{2} + x_2 \right) + t \left( \frac{a+b}{2} - x_2 \right) \).

Considering these relations and by use of the fact that function \( f \) is convex, we obtain
\[
f \left( \frac{a+b}{2} - x_1 \right) = f \left( (1-t) \left( \frac{a+b}{2} - x_2 \right) + t \left( \frac{a+b}{2} + x_2 \right) \right)
\leq (1-t)f \left( \frac{a+b}{2} - x_2 \right) + t f \left( \frac{a+b}{2} + x_2 \right),
\]
so
\[
f \left( \frac{a+b}{2} - x_1 \right) \leq (1-t)f \left( \frac{a+b}{2} - x_2 \right) + t f \left( \frac{a+b}{2} + x_2 \right).
\]

Analogously,
\[
f \left( \frac{a+b}{2} + x_1 \right) \leq (1-t)f \left( \frac{a+b}{2} + x_2 \right) + t f \left( \frac{a+b}{2} - x_2 \right).
\]

Adding the above relations, we obtain
\[
f \left( \frac{a+b}{2} - x_1 \right) + f \left( \frac{a+b}{2} + x_1 \right) \leq f \left( \frac{a+b}{2} - x_2 \right) + f \left( \frac{a+b}{2} + x_2 \right),
\]
that is \( g(x_1) \leq g(x_2) \). So, the function \( g \) is increasing on \( \left[0, \frac{b-a}{2}\right] \). \( \square \)

LEMMA 4. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function on \( [a, b] \) with the property that \( \int_a^b f(x)dx = 0 \). Then there exists \( \alpha \in \left(0, \frac{b-a}{2}\right) \), such that
\[
f \left( \frac{a+b}{2} - \alpha \right) + f \left( \frac{a+b}{2} + \alpha \right) = 0. \tag{2}
\]

Proof. Let \( F : [0, 1] \to \mathbb{R} \) be the function defined by \( F(t) = \int_{(1-t)a+tb}^{(1-t)a+(1-t)b} f(x)dx \).

We have \( F(0) = F \left( \frac{1}{2} \right) = F(1) = 0 \). Since \( f \) is a continuous function, it results that \( F \) is a Rolle function.

Applying Rolle’s theorem to function \( F \) on the intervals \( \left[0, \frac{1}{2}\right] \) and \( \left[\frac{1}{2}, 1\right] \), we obtain that there exists \( c_1 \in \left(0, \frac{1}{2}\right), c_2 \in \left(\frac{1}{2}, 1\right) \) such that
\[
F'(c_1) = F'(c_2) = 0. \tag{3}
\]
We have \( F'(t) = f(ta + (1-t)b)(a-b) - f((1-t)a + tb)(b-a) \), and so we have that \( F'(t) = (a-b)[f(ta + (1-t)b) + f((1-t)a + tb)] \). Then from (3) we have
\[
(4) \quad f(c_k a + (1-c_k)b) + f((1-c_k)a + c_k b) = 0, \quad k \in \{1, 2\}.
\]

Since \( c_k \neq \frac{1}{2} \), \( k \in \{1, 2\} \), it follows that \( c_k a + (1-c_k)b \neq \frac{a+b}{2}, k \in \{1, 2\} \). As an observation, it is possible that \( c_2 = 1 - c_1 \). Consequently, from (4) we have that there exists \( c \in (0, \frac{1}{2}) \) such that
\[
(5) \quad f((ca + (1-c)b) + f((1-c)a + cb) = 0.
\]

We have \((1-c)a + cb < ca + (1-c)b\), since this inequality is equivalent to \(0 < (b-a)(1-2c)\), where \(0 < c < \frac{1}{2}\).

Let \( \alpha = \frac{(b-a)(1-2c)}{2} = \frac{b-a}{2} - c(b-a) \). Since \(0 < c < \frac{1}{2}\), it can be immediately checked that \(0 < \alpha < \frac{b-a}{2}\) and that \( \frac{a+b}{2} - \alpha = (1-c)a + cb, \frac{a+b}{2} + \alpha = ca + (1-c)b \).

Then, considering (5), we have that there exists \( \alpha \in (0, \frac{b-a}{2}) \) such that (2) holds.

**Theorem 5.** If \( f : [a, b] \rightarrow \mathbb{R} \) is a continuous function on interval \([a, b]\), then there exists \( \alpha \in (0, \frac{b-a}{2}) \) so that
\[
(6) \quad \frac{1}{2} \left[ f\left(\frac{a+b}{2} - \alpha\right) + f\left(\frac{a+b}{2} + \alpha\right) \right] = \frac{1}{b-a} \int_a^b f(x)dx.
\]

**Proof.** Let \( g : [a, b] \rightarrow \mathbb{R} \) be the function \( g(x) = f(x) - \frac{1}{b-a} \int_a^b f(t)dt \), \( \forall x \in [a, b] \).

Since the function \( g \) is continuous on interval \([a, b]\), and
\[
\int_a^b g(x)dx = \int_a^b \left[ f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right] dx = \int_a^b f(x)dx - \int_a^b f(t)dt = 0,
\]
are among the conditions of Lemma 4, there exists \( \alpha \in \left(0, \frac{b-a}{2}\right)\), such that \( g\left(\frac{a+b}{2} - \alpha\right) + g\left(\frac{a+b}{2} + \alpha\right) = 0 \). Replacing the function \( g \), we obtain that
\[
\left[ f\left(\frac{a+b}{2} - \alpha\right) - \frac{1}{b-a} \int_a^b f(t)dt \right] + \left[ f\left(\frac{a+b}{2} + \alpha\right) - \frac{1}{b-a} \int_a^b f(t)dt \right] = 0,
\]
and there from comes (6). \( \square \)

**Lemma 6.** Let \( n \in \mathbb{N}, n \geq 2 \). The equation \( \left(\frac{1}{2} - x\right)^n + \left(\frac{1}{2} + x\right)^n = \frac{2}{n+1} \) has one and only one solution on the interval \(\left(0, \frac{1}{2}\right)\).

**Proof.** Let \( f : \left(0, \frac{1}{2}\right) \rightarrow \mathbb{R} \) be the function defined by
\[
f(x) = \left(\frac{1}{2} - x\right)^n + \left(\frac{1}{2} + x\right)^n - \frac{2}{n+1}.
\]
Then
\[ f'(x) = n \left( \frac{1}{2} + x \right)^{n-1} - \left( \frac{1}{2} - x \right)^{n-1} \]
and from the variation of function \( f \), it results that the function \( f \) has only one zero on the interval \( (0, \frac{1}{2}) \). \( \square \)

**Theorem 7.** If the function \( f : [a, b] \to \mathbb{R} \) is continuous on the interval \([a, b]\), then \( \forall x \in (a, b) \), \( \exists c(x) \in \left(0, \frac{b-a}{2}\right) \), such that
\[
\frac{1}{2} \left[ f \left( \frac{a+x}{2} - c(x) \right) + f \left( \frac{a+x}{2} + c(x) \right) \right] = \frac{1}{x-a} \int_a^x f(t)dt.
\]

**Proof.** We apply now Theorem 5 to the restriction of the function \( f \) on the interval \([a, x]\). \( \square \)

**Theorem 8.** We have the function \( f : [a, b] \to \mathbb{R} \) which verifies the conditions:

(i) there exists a neighborhood \( V \) of the point \( a \) so that the function \( f \) is \( n+1 \) times derivable on \( V \cap [a, b] \), and \( f^{(n+1)} \) is bounded on \( V \cap [a, b] \), where \( n \in \mathbb{N} \), \( n \geq 2 \) is fixed;

(ii) \( f^{(n)}(a) = f''^{(n)}(a) = \cdots = f^{(n-1)}(a) = 0 \), \( n \geq 3 \);

(iii) \( f^{(n)}(a) \neq 0 \).

Then, for every \( x \in V \cap (a, b) \), the number \( c(x) \in \left(a, \frac{b-a}{2}\right) \) given by Theorem 7 has the property that there exists \( \lim_{x \to a} \frac{c(x)}{x-a} = l \), \( l \in \left(0, \frac{1}{2}\right) \), and \( l \) is the unique solution of the equation
\[
\left( \frac{1}{2} - l \right)^n + \left( \frac{1}{2} + l \right)^n = \frac{2}{n+1}.
\]

**Proof.** We consider the function \( F : V \cap [a, b] \to \mathbb{R} \) defined by
\[
F(x) = \int_a^x f(t)dt - (x-a)f(a) - \frac{(x-a)^2}{2} f'(a), \quad \forall x \in V \cap [a, b].
\]
We calculate the limit
\[
L = \lim_{x \to a} \frac{F(x)}{(x-a)^{n+1}} = \lim_{x \to a} \frac{(x-a)f(a) - (x-a)f'(a)}{(n+1)(x-a)^n} = \lim_{x \to a} f'(x-a)^{n+1} = \cdots \]
\[
= \frac{1}{(n+1)!} \lim_{x \to a} \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a},
\]
so
\[
L = \lim_{x \to a} \frac{F(x)}{(x-a)^{n+1}} = \frac{1}{(n+1)!} f^{(n)}(a).
\]
According to Taylor’s formula with the rest of Lagrange, for each \( x \in V \cap (a, b) \) there exists \( \xi_1, \xi_2 \),

\[
a < \xi_1 < \frac{a+x}{2} - c(x) < x, \quad a < \xi_2 < \frac{a+x}{2} + c(x) < x,
\]

so that

\[
f \left( \frac{a+x}{2} - c(x) \right) = f(a) + \sum_{k=1}^{n} \frac{\left( \frac{a-x}{2} - c(x) \right)^k}{k!} f^{(k)}(a) + \frac{\left( \frac{a-x}{2} - c(x) \right)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_1)
\]

and

\[
f \left( \frac{a+x}{2} + c(x) \right) = f(a) + \sum_{k=1}^{n} \frac{\left( \frac{a-x}{2} + c(x) \right)^k}{k!} f^{(k)}(a) + \frac{\left( \frac{a-x}{2} + c(x) \right)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_2),
\]

from where, considering (ii) it results that

\[
\frac{1}{2} \left[ f \left( \frac{a+x}{2} - c(x) \right) + f \left( \frac{a+x}{2} + c(x) \right) \right] =
\]

\[
f(a) + \frac{x-a}{2} f'(a) + \frac{1}{2} \left( \frac{x-a}{2} - c(x) \right)^n + \frac{\left( \frac{x-a}{2} + c(x) \right)^n}{n!} f^{(n)}(a) +
\]

\[
+ \frac{1}{2} \frac{\left( \frac{x-a}{2} - c(x) \right)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_1) + \frac{1}{2} \frac{\left( \frac{x-a}{2} + c(x) \right)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_2).
\]

From relation \( 0 < c(x) < \frac{x-a}{2} \), it results that \( \lim_{x \to a} c(x) = 0 \), so

\[
\lim_{x \to a} \left( \frac{x-a}{2} - c(x) \right) = 0, \quad \lim_{x \to a} \left( \frac{x-a}{2} + c(x) \right) = 0,
\]

and \( 0 < \frac{c(x)}{x-a} < \frac{1}{2} \), \( \forall x \in V \cap (a, b) \), from where we obtain that

\[
\frac{1}{2} < \frac{1}{2} + \frac{c(x)}{x-a} < 1, \quad 0 < \frac{1}{2} - \frac{c(x)}{x-a} < \frac{1}{2}, \quad \forall x \in V \cap (a, b).
\]

So the functions \( x \to \frac{1}{2} - \frac{c(x)}{x-a} \), \( \frac{1}{2} + \frac{c(x)}{x-a} \) are bounded on \( V \cap (a, b) \) and considering the condition (i), it results that the functions \( \left( \frac{1}{2} - \frac{c(x)}{x-a} \right)^n f^{(n+1)}(\xi_1) \), \( \left( \frac{1}{2} + \frac{c(x)}{x-a} \right)^n f^{(n+1)}(\xi_2) \) are also bounded on \( V \cap (a, b) \). From these observation and from (11), it results that

\[
\lim_{x \to a} \left( \frac{x-a}{2} - c(x) \right) \left( \frac{1}{2} - \frac{c(x)}{x-a} \right)^n f^{(n+1)}(\xi_1) = 0
\]

and

\[
\lim_{x \to a} \left( \frac{x-a}{2} + c(x) \right) \left( \frac{1}{2} + \frac{c(x)}{x-a} \right)^n f^{(n+1)}(\xi_2) = 0.
\]
Considering \((7)\) and \((10)\), we have
\[
L = \lim_{x \to a} \frac{F(x)}{(x-a)^{n+1}} \\
= \frac{1}{2} \lim_{x \to a} \left( \frac{(x-a) - c(x)}{2} \right)^n \frac{f^{(n)}(a)}{n! (x-a)^n} \\
+ \frac{\left( \frac{x-a}{2} - c(x) \right)^{n+1}}{(n+1)! (x-a)^n} f^{(n+1)}(\xi_1) \\
+ \frac{\left( \frac{x-a}{2} + c(x) \right)^{n+1}}{(n+1)! (x-a)^n} f^{(n+1)}(\xi_2)
\]
and using \((12)\) and \((13)\), we obtain
\[
(14) \quad L = \lim_{x \to a} \frac{F(x)}{(x-a)^{n+1}} = \frac{f^{(n)}(a)}{2n!} \lim_{x \to a} \left[ \left( \frac{1}{2} - \frac{c(x)}{x-a} \right)^n + \left( \frac{1}{2} + \frac{c(x)}{x-a} \right)^n \right].
\]
We shall prove that there exists \(\lim_{x \to a} \frac{c(x)}{x-a}\). Assuming the contrary that this limit does not exist, then there exist two sequences \((x_m)_{m \geq 0}\), \((y_m)_{m \geq 0}\), \(x_m, y_m \in V \cap (a, b)\), \(\forall m \in \mathbb{N}\), so that \(\lim_{m \to \infty} x_m = \lim_{m \to \infty} y_m = a\), \(\lim_{m \to \infty} \frac{c(x_m)}{x_m-a} = l_1 \in (0, \frac{1}{2})\), \(\lim_{m \to \infty} \frac{c(y_m)}{y_m-a} = l_2 \in (0, \frac{1}{2})\) and \(l_1 \neq l_2\).

From (iii), \((9)\) and \((14)\) it results that
\[
\left( \frac{1}{2} - l_1 \right)^n + \left( \frac{1}{2} + l_1 \right)^n = \left( \frac{1}{2} - l_2 \right)^n + \left( \frac{1}{2} + l_2 \right)^n = \frac{2}{n+1},
\]
and considering Lemma 6, we have \(l_1 = l_2\), which is a contradiction.

Since we proved that there exists \(\lim_{x \to a} \frac{c(x)}{x-a}\), from (iii), \((9)\) and \((14)\), we obtain that \(l\) verifies \((8)\). Also considering Lemma 6, Theorem 8 is proved. \(\square\)

**Corollary 9.** In the conditions of Theorem 8, for \(n = 2\), we obtain that \(l = \lim_{x \to a} \frac{c(x)}{x-a} = \frac{1}{2\sqrt{3}}\), and for \(n = 4\), we obtain that \(l = \lim_{x \to a} \frac{c(x)}{x-a} = \frac{\sqrt{2\sqrt{3} - 15}}{20}\).

**Example.** Let \(0 < a\) and \(f : [a, b] \to \mathbb{R}\) be the function \(f(x) = \frac{1}{x-a}\). According to Theorem 7, \(\forall x \in (a, b)\), \(\exists \ c(x) \in (0, \frac{x-a}{2})\) which verifies \((7)\), and from this we obtain that
\[
c(x) = \sqrt{\left( \frac{x+a}{2} \right)^2 - \frac{x^2-a^2}{2}}.
\]
According to Corollary 9, \(\lim_{x \to a} \frac{c(x)}{x-a} = \frac{1}{2\sqrt{3}}\). \(\square\)
THEOREM 10. Let $f : [a, b] \to \mathbb{R}$ be a convex function, $f$ is continuous on the right at $a$ and continuous on the left at $b$. Then there exists $\alpha \in \left(0, \frac{b-a}{2}\right)$ so that $\forall x \in [0, \alpha]$, $\forall y \in \left[\alpha, \frac{b-a}{2}\right]$, we have

\begin{equation}
\frac{1}{2} \left[ f \left( \frac{a+b}{2} - x \right) + f \left( \frac{a+b}{2} + x \right) \right] \\
\leq \frac{1}{2} \left[ f \left( \frac{a+b}{2} - \alpha \right) + f \left( \frac{a+b}{2} + \alpha \right) \right] \\
= \frac{1}{b-a} \int_{a}^{b} f(t)dt \\
\leq \frac{1}{2} \left[ f \left( \frac{a+b}{2} - y \right) + f \left( \frac{a+b}{2} + y \right) \right] \\
\leq \frac{1}{2} \left[ f(a) + f(b) \right].
\end{equation}

Proof. Since $f$ is a convex function on interval $[a, b]$, it results that $f$ is continuous on $(a, b)$, and since $f$ is continuous on the right at $a$ and continuous on the left at $b$, we know that $f$ is continuous on interval $[a, b]$. Next Theorem 3 and Theorem 5 are to be applied.

Remarks. Theorem 10 is an extension and refinement of Hermite-Hadamard’s inequality.

Next, we will show that the maximal value of $\alpha$ with the property from Theorem 5, is $\alpha_{\text{max}} = \frac{b-a}{2}$.

EXAMPLE. Let $f : [-1, 1] \to \mathbb{R}$ be the function defined by

\[
f_c(x) = \begin{cases} 
-\frac{1}{1-c} x - \frac{1}{c} & , \quad x \in [-1, -c) \\
0 & , \quad x \in [-c, c] \\
\frac{1}{1-c} x - \frac{c}{1-c} & , \quad x \in (c, 1],
\end{cases}
\]

where $c \in [0, 1)$.

We have that $\frac{1}{2} \int_{-1}^{1} f_c(x)dx = \frac{1-c}{2}$, and $\frac{1}{2} [f_c(-x) + f_c(x)] = 0$, $\forall x \in [0, c]$.

We determine $\alpha$ on interval $(c, 1)$. We have $\frac{1}{2} [f_c(-\alpha) + f_c(\alpha)] = \int_{-1}^{1} f_c(t)dt$, equivalent to $\frac{\alpha}{1-c} - \frac{c}{1-c} = \frac{1-c}{2}$, from which we have $\alpha = \frac{1+c}{2}$.

If $c$ tended towards 1, than $\alpha$ would tend 1 (that is $\frac{b-a}{2}$, where $a = -1$ and $b = 1$), so the maximal value of $\alpha$ with the property from Theorem 10 is $\alpha = \frac{b-a}{2}$.

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