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ON A MEAN VALUE THEOREM CONNECTED WITH HERMITE-HADAMARD'S INEQUALITY

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Abstract. In this paper we prove a mean-value theorem for integral calculus, then we demonstrate properties of the mean point. In the end we give an extension of Hermite-Hadamard's inequality.

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1. INTRODUCTION

In this article, we will give two interpolations of the Hermite-Hadamard's inequality. We start from:

THEOREM 1 (Hermite-Hadamard). *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then*

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a} \int_b^a f(x)\mathrm{d}x \leq \tfrac{f(a)+f(b)}{2}\,. \tag{1}$$

For the proofs we refer to [3] or [8].

2. PRELIMINARIES

It is known that if $a < b$, then $x \in [a, b]$ iff there exists $t \in [0, 1]$ such that $x = (1 - t)a + tb$, and if $x_1, x_2 \in [a, b]$, $x_1 \leq x_2$ are symmetrical with respect to the middle point of the interval $[a, b]$ iff there exists $c \in \left[0, \frac{b-a}{2}\right]$ such that $x_1 = \frac{a+b}{2} - c$ and $x_2 = \frac{a+b}{2} + c$.

LEMMA 2. *Let $[a, b]$, $a < b$, be an interval. The points $x_1, x_2 \in [a, b]$, $x_1 \leq x_2$ are symmetrical with respect to the middle point of the interval $[a, b]$ iff there exists $t \in \left[0, \frac{1}{2}\right]$, such that $x_1 = (1 - t)a + tb$, and $x_2 = (1 - t)b + ta$.*

Proof. Let $t = \frac{\frac{b-a}{2} - c}{b-a} = \frac{1}{2} - \frac{c}{b-a}$. It can be checked that $t \in \left[0, \frac{1}{2}\right]$, and that $x_1 = (1 - t)a + tb$, $x_2 = (1 - t)b + ta$. Reciprocally, it is proved that $\frac{x_1+x_2}{2} = \frac{a+b}{2}$. □

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3. MAIN RESULTS

THEOREM 3. *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then the function*

$$g:\left[0,\frac{b-a}{2}\right]\to\mathbb{R},\ g(x)=f\left(\frac{a+b}{2}-x\right)+f\left(\frac{a+b}{2}+x\right),\ \forall\, x\in\left[0,\frac{b-a}{2}\right]$$

is increasing on $\left[0,\frac{b-a}{2}\right]$.

Proof. Let $0\le x_1<x_2\le\frac{b-a}{2}$. Then

$$a\le\frac{a+b}{2}-x_2<\frac{a+b}{2}-x_1\le\frac{a+b}{2}+x_1<\frac{a+b}{2}+x_2\le b,$$

and considering Lemma 2, we find that there exists $t\in(0,\frac{1}{2})$ so that $\frac{a+b}{2}-x_1=(1-t)\left(\frac{a+b}{2}-x_2\right)+t\left(\frac{a+b}{2}+x_2\right)$, and $\frac{a+b}{2}+x_1=(1-t)\left(\frac{a+b}{2}+x_2\right)+t\left(\frac{a+b}{2}-x_2\right)$.

Considering these relations and by use of the fact that function f is convex, we obtain

$$\begin{aligned}f\left(\frac{a+b}{2}-x_1\right)&=f\left((1-t)\left(\frac{a+b}{2}-x_2\right)+t\left(\frac{a+b}{2}+x_2\right)\right)\\&\le(1-t)f\left(\frac{a+b}{2}-x_2\right)+t\,f\left(\frac{a+b}{2}+x_2\right),\end{aligned}$$

so

$$f\left(\frac{a+b}{2}-x_1\right)\le(1-t)f\left(\frac{a+b}{2}-x_2\right)+t\,f\left(\frac{a+b}{2}+x_2\right).$$

Analogously,

$$f\left(\frac{a+b}{2}+x_1\right)\le(1-t)f\left(\frac{a+b}{2}+x_2\right)+t\,f\left(\frac{a+b}{2}-x_2\right).$$

Adding the above relations, we obtain

$$f\left(\frac{a+b}{2}-x_1\right)+f\left(\frac{a+b}{2}+x_1\right)\le f\left(\frac{a+b}{2}-x_2\right)+f\left(\frac{a+b}{2}+x_2\right),$$

that is $g(x_1)\le g(x_2)$. So, the function g is increasing on $\left[0,\frac{b-a}{2}\right]$. □

LEMMA 4. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$ with the property that $\int_a^b f(x)\mathrm{d}x=0$. Then there exists $\alpha\in\left(0,\frac{b-a}{2}\right)$, such that*

$$f\left(\frac{a+b}{2}-\alpha\right)+f\left(\frac{a+b}{2}+\alpha\right)=0\,. \tag{2}$$

Proof. Let $F:[0,1]\to\mathbb{R}$ be the function defined by $F(t)=\int_{(1-t)a+tb}^{ta+(1-t)b}f(x)\mathrm{d}x$.

We have $F(0)=F\left(\frac{1}{2}\right)=F(1)=0$. Since f is a continuous function, it results that F is a Rolle function.

Applying Rolle's theorem to function F on the intervals $\left[0,\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$, we obtain that there exists $c_1\in\left(0,\frac{1}{2}\right)$, $c_2\in\left(\frac{1}{2},1\right)$ such that

$$F'(c_1)=F'(c_2)=0\,. \tag{3}$$

We have $F'(t) = f\big(ta + (1-t)b\big)(a-b) - f\big((1-t)a + tb\big)(b-a)$, and so we have that $F'(t) = (a-b)\big[f\big(ta + (1-t)b\big) + f\big((1-t)a + tb\big)\big]$.
Then from (3) we have

$$f\big(c_k a + (1-c_k)b\big) + f\big((1-c_k)a + c_k b\big) = 0\,, \quad k \in \{1,2\}\,. \tag{4}$$

Since $c_k \neq \frac{1}{2}\,,\ k \in \{1,2\}$, it follows that $c_k a + (1-c_k)b \neq \frac{a+b}{2}\,,\ k \in \{1,2\}$. As an observation, it is possible that $c_2 = 1 - c_1$. Consequently, from (4) we have that there exists $c \in \left(0, \frac{1}{2}\right)$ such that

$$f\big(ca + (1-c)b\big) + f\big((1-c)a + cb\big) = 0\,. \tag{5}$$

We have $(1-c)a + cb < ca + (1-c)b$, since this inequality is equivalent to $0 < (b-a)(1-2c)$, where $0 < c < \frac{1}{2}\,$.

Let $\alpha = \frac{(b-a)(1-2c)}{2} = \frac{b-a}{2} - c(b-a)$. Since $0 < c < \frac{1}{2}$, it can be immediately checked that $0 < \alpha < \frac{b-a}{2}$ and that $\frac{a+b}{2} - \alpha = (1-c)a + cb$, $\frac{a+b}{2} + \alpha = ca + (1-c)b$.

Then, considering (5), we have that there exists $\alpha \in \left(0, \frac{b-a}{2}\right)$ such that (2) holds. □

THEOREM 5. *If $f : [a,b] \to \mathbb{R}$ is a continuous function on interval $[a,b]$, then there exists $\alpha \in \left(0, \frac{b-a}{2}\right)$ so that*

$$\tfrac{1}{2}\left[f\left(\tfrac{a+b}{2} - \alpha\right) + f\left(\tfrac{a+b}{2} + \alpha\right)\right] = \tfrac{1}{b-a}\int_a^b f(x)\mathrm{d}x\,. \tag{6}$$

Proof. Let $g : [a,b] \to \mathbb{R}$ be the function $g(x) = f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$, $\forall\, x \in [a,b]$.

Since the function g is continuous on interval $[a,b]$, and

$$\int_a^b g(x)\mathrm{d}x = \int_a^b \left[f(x) - \tfrac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right]\mathrm{d}x = \int_a^b f(x)\mathrm{d}x - \int_a^b f(t)\mathrm{d}t = 0,$$

are among the conditions of Lemma 4, there exists $\alpha \in \left(0, \frac{b-a}{2}\right)$, such that $g\left(\frac{a+b}{2} - \alpha\right) + g\left(\frac{a+b}{2} + \alpha\right) = 0$. Replacing the function g, we obtain that

$$\left[f\left(\tfrac{a+b}{2} - \alpha\right) - \tfrac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right] + \left[f\left(\tfrac{a+b}{2} + \alpha\right) - \tfrac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right] = 0,$$

and there from comes (6). □

LEMMA 6. *Let $n \in \mathbb{N}$, $n \geq 2$. The equation $\left(\frac{1}{2} - x\right)^n + \left(\frac{1}{2} + x\right)^n = \frac{2}{n+1}$ has one and only one solution on the interval $\left(0, \frac{1}{2}\right)$.*

Proof. Let $f : \left(0, \frac{1}{2}\right) \to \mathbb{R}$ be the function defined by

$$f(x) = \left(\tfrac{1}{2} - x\right)^n + \left(\tfrac{1}{2} + x\right)^n - \tfrac{2}{n+1}\,.$$

Then

$$f'(x) = n\left[\left(\tfrac{1}{2}+x\right)^{n-1} - \left(\tfrac{1}{2}-x\right)^{n-1}\right]$$

and from the variation of function f, it results that the function f has only one zero on the interval $\left(0,\frac{1}{2}\right)$. □

THEOREM 7. *If the function $f:[a,b]\to\mathbb{R}$ is continuous on the interval $[a,b]$, then $\forall x\in(a,b]$, $\exists\, c(x)\in\left(0,\frac{x-a}{2}\right)$, such that*

$$\tfrac{1}{2}\left[f\left(\tfrac{a+x}{2}-c(x)\right)+f\left(\tfrac{a+x}{2}+c(x)\right)\right]=\tfrac{1}{x-a}\int_a^x f(t)\mathrm{d}t. \tag{7}$$

Proof. We apply now Theorem 5 to the restriction of the function f on the interval $[a,x]$. □

THEOREM 8. *We have the function $f:[a,b]\to\mathbb{R}$ which verifies the conditions:*

(i) *there exists a neighborhood V of the point a so that the function f is $n+1$ times derivable on $V\cap[a,b]$, and $f^{(n+1)}$ is bounded on $V\cap[a,b]$, where $n\in\mathbb{N}$, $n\geq 2$ is fixed;*
(ii) $f''(a)=f'''(a)=\cdots=f^{(n-1)}(a)=0$, $n\geq 3$;
(iii) $f^{(n)}(a)\neq 0$.

Then, for every $x\in V\cap(a,b]$, the number $c(x)\in\left(a,\frac{x-a}{2}\right)$ given by Theorem 7 has the property that there exists $\lim_{x\searrow a}\frac{c(x)}{x-a}=l$, $l\in\left(0,\frac{1}{2}\right)$, and l is the unique solution of the equation

$$\left(\tfrac{1}{2}-l\right)^n+\left(\tfrac{1}{2}+l\right)^n=\tfrac{2}{n+1}\,. \tag{8}$$

Proof. We consider the function $F:V\cap[a,b]\to\mathbb{R}$ defined by

$$F(x)=\int_a^x f(t)\mathrm{d}t-(x-a)f(a)-\tfrac{(x-a)^2}{2}\,f'(a)\,,\quad \forall x\in V\cap[a,b]\,.$$

We calculate the limit

$$\begin{aligned}L&=\lim_{\substack{x\to a\\x>a}}\tfrac{F(x)}{(x-a)^{n+1}}\\&=\lim_{\substack{x\to a\\x>a}}\tfrac{f(x)-f(a)-(x-a)f'(a)}{(n+1)(x-a)^n}\\&=\tfrac{1}{n+1}\lim_{\substack{x\to a\\x>a}}\tfrac{f'(x)-f'(a)}{n(x-a)^{n-1}}=\cdots\\&=\tfrac{1}{(n+1)\cdot n\cdot\ldots\cdot 2}\lim_{\substack{x\to a\\x>a}}\tfrac{f^{(n-1)}(x)-f^{(n-1)}(a)}{x-a}\,,\end{aligned}$$

so

$$L=\lim_{\substack{x\to a\\x>a}}\tfrac{F(x)}{(x-a)^{n+1}}=\tfrac{1}{(n+1)!}\,f^{(n)}(a)\,. \tag{9}$$

According to Taylor's formula with the rest of Lagrange, for each $x \in V \cap (a,b]$ there exists ξ_1, ξ_2,

$$a < \xi_1 < \tfrac{a+x}{2} - c(x) < x, \quad a < \xi_2 < \tfrac{a+x}{2} + c(x) < x\,,$$

so that

$$f\left(\tfrac{a+x}{2} - c(x)\right) = f(a) + \sum_{k=1}^{n} \frac{\left(\frac{x-a}{2} - c(x)\right)^k}{k!} f^{(k)}(a) + \frac{\left(\frac{x-a}{2} - c(x)\right)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_1)$$

and

$$f\left(\tfrac{a+x}{2} + c(x)\right) = f(a) + \sum_{k=1}^{n} \frac{\left(\frac{x-a}{2} + c(x)\right)^k}{k!} f^{(k)}(a) + \frac{\left(\frac{x-a}{2} + c(x)\right)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_2)\,,$$

from where, considering (ii) it results that

$$(10) \qquad \begin{aligned} &\tfrac{1}{2}\left[f\left(\tfrac{a+x}{2} - c(x)\right) + f\left(\tfrac{a+x}{2} + c(x)\right)\right] = \\ &= f(a) + \tfrac{x-a}{2} f'(a) + \tfrac{1}{2} \frac{\left(\frac{x-a}{2} - c(x)\right)^n + \left(\frac{x-a}{2} + c(x)\right)^n}{n!} f^{(n)}(a) + \\ &+ \tfrac{1}{2} \frac{\left(\frac{x-a}{2} - c(x)\right)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_1) + \tfrac{1}{2} \frac{\left(\frac{x-a}{2} + c(x)\right)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_2)\,. \end{aligned}$$

From relation $0 < c(x) < \frac{x-a}{2}$, it results that $\lim\limits_{\substack{x \to a \\ x > a}} c(x) = 0$, so

$$(11) \qquad \lim_{\substack{x \to a \\ x > a}} \left(\tfrac{x-a}{2} - c(x)\right) = 0\,, \quad \lim_{\substack{x \to a \\ x > a}} \left(\tfrac{x-a}{2} + c(x)\right) = 0\,,$$

and $0 < \frac{c(x)}{x-a} < \frac{1}{2}$, $\forall x \in V \cap (a,b]$, from where we obtain that $\frac{1}{2} < \frac{1}{2} + \frac{c(x)}{x-a} < 1$, $0 < \frac{1}{2} - \frac{c(x)}{x-a} < \frac{1}{2}$, $\forall x \in V \cap (a,b]$.

So the functions $x \to \frac{1}{2} - \frac{c(x)}{x-a}$, $\frac{1}{2} + \frac{c(x)}{x-a}$ are bounded on $V \cap (a,b]$ and considering the condition (i), it results that the functions $\left(\frac{1}{2} - \frac{c(x)}{x-a}\right)^n f^{(n+1)}(\xi_1)$, $\left(\frac{1}{2} + \frac{c(x)}{x-a}\right)^n f^{(n+1)}(\xi_2)$ are also bounded on $V \cap (a,b]$. From these observation and from (11), it results that

$$(12) \qquad \lim_{\substack{x \to a \\ x > a}} \left(\tfrac{x-a}{2} - c(x)\right)\left(\tfrac{1}{2} - \tfrac{c(x)}{x-a}\right)^n f^{(n+1)}\xi_1) = 0$$

and

$$(13) \qquad \lim_{\substack{x \to a \\ x > a}} \left(\tfrac{x-a}{2} + c(x)\right)\left(\tfrac{1}{2} + \tfrac{c(x)}{x-a}\right)^n f^{(n+1)}(\xi_2) = 0\,.$$

Considering (7) and (10), we have

$$L = \lim_{\substack{x\to a\\ x>a}} \frac{F(x)}{(x-a)^{n+1}}$$

$$= \frac{1}{2}\lim_{\substack{x\to a\\ x>a}} \frac{\frac{\left(\frac{x-a}{2}-c(x)\right)^n+\left(\frac{x-a}{2}+c(x)\right)^n}{n!}f^{(n)}(a)+}{(x-a)^n}$$

$$\frac{+\frac{\left(\frac{x-a}{2}-c(x)\right)^{n+1}}{(n+1)!}f^{(n+1)}(\xi_1)+\frac{\left(\frac{x-a}{2}+c(x)\right)^{n+1}}{(n+1)!}f^{(n+1)}(\xi_2)}{(x-a)^n} =$$

$$= \frac{1}{2}\lim_{\substack{x\to a\\ x>a}} \left[\frac{\left(\frac{1}{2}-\frac{c(x)}{x-a}\right)^n+\left(\frac{1}{2}+\frac{c(x)}{x-a}\right)^n}{n!}f^{(n)}(a)\right.$$

$$+\frac{1}{(n+1)!}\left(\frac{x-a}{2}-c(x)\right)\left(\frac{1}{2}-\frac{c(x)}{x-a}\right)^n f^{(n+1)}(\xi_1)$$

$$\left.+\frac{1}{(n+1)!}\left(\frac{x-a}{2}+c(x)\right)\left(\frac{1}{2}+\frac{c(x)}{x-a}\right)^n f^{(n+1)}(\xi_2)\right]$$

and using (12) and (13), we obtain

$$L=\lim_{\substack{x\to a\\ x>a}}\frac{F(x)}{(x-a)^{n+1}}=\frac{f^{(n)}(a)}{2n!}\lim_{\substack{x\to a\\ x>a}}\left[\left(\frac{1}{2}-\frac{c(x)}{x-a}\right)^n+\left(\frac{1}{2}+\frac{c(x)}{x-a}\right)^n\right]. \tag{14}$$

We shall prove that there exists $\lim_{x\searrow a}\frac{c(x)}{x-a}$. Assuming the contrary that this limit does not exist, then there exist two sequences $(x_m)_{m\geq 0}$, $(y_m)_{m\geq 0}$, $x_m, y_m \in V\cap(a,b]$, $\forall m\in\mathbb{N}$, so that $\lim_{m\to\infty}x_m = \lim_{m\to\infty}y_m = a$, $\lim_{m\to\infty}\frac{c(x_m)}{x_m-a}=l_1\in\left(0,\frac{1}{2}\right)$, $\lim_{m\to\infty}\frac{c(y_m)}{y_m-a}=l_2\in\left(0,\frac{1}{2}\right)$ and $l_1\neq l_2$.

From (iii), (9) and (14) it results that

$$\left(\frac{1}{2}-l_1\right)^n+\left(\frac{1}{2}+l_1\right)^n=\left(\frac{1}{2}-l_2\right)^n+\left(\frac{1}{2}+l_2\right)^n=\frac{2}{n+1},$$

and considering Lemma 6, we have $l_1=l_2$, which is a contradiction.

Since we proved that there exists $\lim_{x\searrow a}\frac{c(x)}{x-a}$, from (iii), (9) and (14), we obtain that l verifies (8). Also considering Lemma 6, Theorem 8 is proved. □

Corollary 9. *In the conditions of Theorem* 8, *for* $n=2$, *we obtain that* $l=\lim_{x\searrow a}\frac{c(x)}{x-a}=\frac{1}{2\sqrt{3}}$, *and for* $n=4$, *we obtain that* $l=\lim_{x\searrow a}\frac{c(x)}{x-a}=\sqrt{\frac{2\sqrt{70}-15}{20}}$.

Example. Let $0<a$ and $f:[a,b]\to\mathbb{R}$ be the function $f(x)=\frac{1}{x}$. According to Theorem 7, $\forall x\in(a,b]$, $\exists\, c(x)\in\left(0,\frac{x-a}{2}\right)$ which verifies (7), and from this we obtain that

$$c(x)=\sqrt{\left(\frac{x+a}{2}\right)^2-\frac{x^2-a^2}{2\ln\frac{x}{a}}}.$$

According to Corollary 9, $\lim_{x\searrow a}\frac{c(x)}{x-a}=\frac{1}{2\sqrt{3}}$. □

THEOREM 10. *Let $f : [a,b] \to \mathbb{R}$ be a convex function, f is continuous on the right at a and continuous on the left at b. Then there exists $\alpha \in \left(0, \frac{b-a}{2}\right)$ so that $\forall x \in [0,\alpha]$, $\forall y \in \left[\alpha, \frac{b-a}{2}\right]$, we have*

$$
\begin{aligned}
f\left(\tfrac{a+b}{2}\right) &\le \tfrac{1}{2}\left[f\left(\tfrac{a+b}{2}-x\right)+f\left(\tfrac{a+b}{2}+x\right)\right] \\
&\le \tfrac{1}{2}\left[f\left(\tfrac{a+b}{2}-\alpha\right)+f\left(\tfrac{a+b}{2}+\alpha\right)\right] \\
&= \tfrac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\
&\le \tfrac{1}{2}\left[f\left(\tfrac{a+b}{2}-y\right)+f\left(\tfrac{a+b}{2}+y\right)\right] \\
&\le \tfrac{1}{2}\left[f(a)+f(b)\right].
\end{aligned} \tag{15}
$$

Proof. Since f is a convex function on interval $[a,b]$, it results that f is continuous on (a,b), and since f is continuous on the right at a and continuous on the left at b, we know that f is continuous on interval $[a,b]$. Next Theorem 3 and Theorem 5 are to be applied. □

REMARKS. Theorem 10 is an extension and refinement of Hermite-Hadamard's inequality. □

Next, we will show that the maximal value of α with the property from Theorem 5, is $\alpha_{\max} = \frac{b-a}{2}$.

EXAMPLE. Let $f : [-1,1] \to \mathbb{R}$ be the function defined by

$$
f_c(x) = \begin{cases} -\frac{1}{1-c}\,x - \frac{c}{1-c}\,, & x \in [-1,-c) \\ 0\,, & x \in [-c,c] \\ \frac{1}{1-c}\,x - \frac{c}{1-c}\,, & x \in (c,1], \end{cases}
$$

where $c \in [0,1)$.

We have that $\frac{1}{2}\int_{-1}^{1} f_c(x)\mathrm{d}x = \frac{1-c}{2}$, and $\frac{1}{2}\left[f_c(-x)+f_c(x)\right] = 0$, $\forall x \in [0,c]$.

We determine α on interval $(c,1)$. We have $\frac{1}{2}\left[f_c(-\alpha)+f_c(\alpha)\right] = \int_{-1}^{1} f_c(t)\mathrm{d}t$, equivalent to $\frac{\alpha}{1-c} - \frac{c}{1-c} = \frac{1-c}{2}$, from which we have $\alpha = \frac{1+c^2}{2}$.

If c tended towards 1, than α would tend 1 (that is $\frac{b-a}{2}$, where $a = -1$ and $b = 1$), so the maximal value of α with the property from Theorem 10 is $\alpha = \frac{b-a}{2}$. □

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