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A NOTE ON THE SOLVABILITY OF THE NONLINEAR WAVE EQUATION

RADU PRECUP*

Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. New existence and localization results for the nonlinear wave equation are established by means of the Schauder fixed point theorem. The main idea is to handle two equivalent operator forms of the wave equation, one of fixed point type giving the operator to which the Schauder theorem applies and an other one of coincidence type for the localization of a solution.

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1. INTRODUCTION

We shall discuss the existence and localization of solutions for the nonlinear problem

$$\text{(1)} \qquad \begin{cases} -u''(t) + \Delta u(t) - mu(t) = f(t, u(t)), & t \in [0,T] \\ u(0) = u(T) = 0 \\ u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^2\left([0,T]; H^{-1}(\Omega)\right). \end{cases}$$

Here $0 < T < \infty$, $\Omega \subset \mathbb{R}^n$ is a bounded open subset, $m > -\lambda_1$ (λ_1 is the first eigenvalue corresponding to $-\Delta$ and to the homogenous Dirichlet boundary condition) and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$.

Here are some notations which will be used in what follows. For a bounded and open set $\Omega \subset \mathbb{R}^n$, $1 \leq p < \infty$ and $0 < T < \infty$, we consider the space $L^p(\Omega)$ with norm $|u|_p = \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p}$ and the space $C([0,T]; L^p(\Omega))$ with norm $|.|_{\infty,p}$ defined by

$$|u|_{\infty,p} = \max_{t \in [0,T]} |u(t)|_p .$$

The space $H^{-1}(\Omega)$ is the dual of the Sobolev space $H_0^1(\Omega)$. We recall that $H_0^1(\Omega) \subset L^p(\Omega)$ and $L^q(\Omega) \subset H^{-1}(\Omega)$ (with continuous imbeddings) for

*Department of Applied Mathematics, "Babeş–Bolyai" University, Cluj, Romania, e-mail: r.precup@math.ubbcluj.ro.

$1 \le p \le 2^* = \frac{2n}{n-2}$ and $q \ge (2^*)' = \frac{2n}{n+2}$ if $n \ge 3$ and for all $p, q \ge 1$ if $n = 1$ or $n = 2$.

Let $A : D(A) \to C([0,T]; H^{-1}(\Omega))$ be given by

$$(Au)(t) = -u''(t).$$

Here $D(A) = \left\{u \in C^2([0,T]; H^{-1}(\Omega)) : u(0) = u(T) = 0\right\}.$

Clearly A is invertible and

$$\left(A^{-1}v\right)(t) = \int_0^T g(t,s)\, v(s)\, ds, \quad v \in C\left([0,T]; H^{-1}(\Omega)\right)$$

where g is the Green's function of the differential operator A with respect to the boundary condition $u(0) = u(T) = 0$, i.e.

$$g(t,s) = \begin{cases} \frac{s(T-t)}{T}, & 0 \le s \le t \le T \\ \frac{t(T-s)}{T}, & 0 \le t \le s \le T. \end{cases}$$

Notice that for every subinterval $[a,b]$ of $[0,T]$, $0 < a < b < T$, g satisfies the following upper and lower inequalities

$$\begin{aligned} g(t,s) &\le g(s,s) \quad \text{for } t \in [0,T] \text{ and } s \in [0,T] \\ k_{a,b}\, g(s,s) &\le g(t,s) \quad \text{for } t \in [a,b] \text{ and } s \in [0,T]. \end{aligned} \tag{2}$$

Here $k_{a,b} = \min\left\{\frac{a}{T}, \ \frac{T-b}{T}\right\}$. Obviously $0 < k_{a,b} < 1$. In what follows we shall also use the notation

$$g^*_{a,b} = \max_{t \in [0,T]} \int_a^b g(t,s)\, ds.$$

Clearly $g^*_{a,b} \ge k_{a,b} \int_a^b g(s,s)\, ds > 0$.

Let $B : H_0^1(\Omega) \to H^{-1}(\Omega)$ be defined by

$$Bu = -\Delta u + m\, u, \quad u \in H_0^1(\Omega).$$

Since $m > -\lambda_1$, B is invertible and its inverse B^{-1} is a linear continuous and positive (by the maximum principle) operator.

Basic theory on the non-homogenous linear wave equation (see [2] and [4]) guarantees that the operator $A - B$ from $C([0,T]; H_0^1(\Omega)) \cap D(A)$ to $C([0,T]; H^{-1}(\Omega))$ is invertible and its inverse $(A-B)^{-1}$ is a linear operator, completely continuous from $C([0,T]; H^{-1}(\Omega))$ to $C([0,T];\ L^p(\Omega))$ for $(2^*)' \le p < 2^*$ if $n \ge 3$ and any $p \ge 1$ if $n = 1$ or $n = 2$.

One can check that the following equality is true

$$\left(B^{-1} - A^{-1}\right)^{-1} = (A-B)^{-1} BA$$

for operators acting from $C^2([0,T]; H_0^1(\Omega))$ to the space $C([0,T]; H_0^1(\Omega)) \cap C^2([0,T];\ H^{-1}(\Omega))$.

Let $F : C([0,T]; H_0^1(\Omega)) \to C([0,T]; H^{-1}(\Omega))$ be defined by

$$F(u)(t) = f(t, u(t)).$$

Now solving (1) is equivalent to the problem

$$(A-B)\,u=F\,(u)\,,\quad u\in C\left([0,T]\,;H_0^1\,(\Omega)\right)\cap D\,(A)$$

which can be written under the form

$$u=(A-B)^{-1}\,F\,(u) \tag{3}$$

or equivalently as

$$\left(B^{-1}-A^{-1}\right)u=A^{-1}B^{-1}F\,(u)\,. \tag{4}$$

Under suitable conditions on F, the complete continuity of $(A-B)^{-1}$ implies that the nonlinear operator $N:=(A-B)^{-1}\,F$ associated to the right hand side of equation (3) is completely continuous. Hence equation (3) gives us the operator to which Schauder's Theorem applies. On the other hand, the upper and lower inequalities (2) for the Green's kernel in A^{-1} make equation (4) useful for the localization of a solution of (3).

2. NONLINEARITIES WITH SUBLINEAR GROWTH

The first result is concerned with nonlinearities $f\,(t,u)$ having a sublinear growth in u.

Theorem 1. *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be continuous with a sublinear growth in its second variable, i.e.*

$$|f\,(t,u)|\leq c+d\,|u|^{\gamma}\,,\;\;t\in[0,T]\,,\;u\in\mathbb{R} \tag{5}$$

for some $c,d>0$ and $\gamma>0$ with $\frac{n-2}{n+2}\leq\gamma<1$. Then problem (1) *has a least one solution. If in addition, $f\geq 0$ and there exists an interval $[a,b]$ with $0<a<b<T$ and a number σ with*

$$0<\sigma\leq f\,(t,u)\,,\;\;t\in[a,b]\,,\;u\in\mathbb{R} \tag{6}$$

then problem (1) *has at least one solution u such that*

$$\left|\left(B^{-1}-A^{-1}\right)u\right|_{\infty,q}\geq\frac{\sigma\,g^*_{a,b}\,|\varphi_1|_q}{(\lambda_1+m)\,|\varphi_1|_\infty}\,. \tag{7}$$

Here $q=2^*$ if $n\geq 3$ and q is any number $\geq\gamma^{-1}$ if $n=1$ or $n=2$, and $\varphi_1>0$ is an eigenfunction of $-\Delta$ corresponding to the first eigenvalue λ_1.

Proof. We shall apply Schauder fixed point theorem (see [1] and [3]). Let $p=\gamma q$. Notice that for $n\geq 3$, from $\frac{n-2}{n+2}\leq\gamma<1$ we have $(2^*)'\leq p=\gamma 2^*<2^*$.

Let $F:C\,([0,T]\,;L^p\,(\Omega))\to C\,([0,T]\,;L^q\,(\Omega))$ be given by

$$F\,(u)\,(t)=f\,(t,u\,(t))\,,\;\;u\in C\,([0,T]\,;L^p\,(\Omega))\,,\;t\in[0,T]\,.$$

From (5) we immediately see that F is well defined from $C\,([0,T]\,;L^p\,(\Omega))$ to $C\,([0,T]\,;L^q\,(\Omega))$ and

$$|F\,(u)\,(t)|_q\leq c^*+d\,|u\,(t)|_p^{\gamma}\,,\;\;t\in[0,T]\,. \tag{8}$$

Here $c^* = c\,|1|_q$. Consequently, F sends bounded subsets of $C([0,T];L^p(\Omega))$ into bounded sets of $C([0,T];L^q(\Omega))$. Also the continuity of f guarantees that F is continuous. It follows that N is completely continuous from the space $C([0,T];L^p(\Omega))$ to itself.

Also, if we let $|\,(A-B)^{-1}\,|$ be the norm of the operator $(A-B)^{-1}$ from $C([0,T];L^q(\Omega))$ to $C([0,T];L^p(\Omega))$ and we use (8), we obtain

$$|N(u)\,|_{\infty,p} \le |\,(A-B)^{-1}\,|(c^* + d\,|u|_{\infty,p}^{\gamma}). \tag{9}$$

Hence, for $u \in C([0,T];L^p(\Omega))$ with $|u|_{\infty,p} \le R$, we have

$$|N(u)|_{\infty,p} \le |\,(A-B)^{-1}\,|\,(c^* + dR^{\gamma}).$$

Now since $\gamma < 1$, we can choose $R > 0$ sufficiently large that

$$|\,(A-B)^{-1}\,|\,(c^* + dR^{\gamma}) \le R.$$

Thus

$$|u|_{\infty,p} \le R \ \text{ implies } \ |N(u)|_{\infty,p} \le R \tag{10}$$

and so Schauder's fixed point theorem applies proving the existence of a solution.

Assume that the additional hypothesis is satisfied. It is well known that any eigenfunction of $-\Delta$ is bounded on Ω, so $0 < \varphi_1(x) \le |\varphi_1|_\infty < \infty$ for all $x \in \Omega$. Then

$$0 < \tfrac{\sigma}{|\varphi_1|_\infty}\,\varphi_1(x) \le \sigma, \quad x \in \Omega.$$

This, together with $-\Delta\varphi_1 + m\varphi_1 = (\lambda_1 + m)\,\varphi_1$ and the positivity of B^{-1}, guarantees that

$$B^{-1}\sigma \ge \tfrac{\sigma}{|\varphi_1|_\infty}\,B^{-1}\varphi_1 = \tfrac{\sigma}{|\varphi_1|_\infty(\lambda_1+m)}\,\varphi_1.$$

As a result

$$\left|B^{-1}\sigma\right|_q \ge \tfrac{\sigma}{|\varphi_1|_\infty(\lambda_1+m)}\,|\varphi_1|_q\,. \tag{11}$$

Let u be a solution and let $t^* \in [0,T]$ be such that $g^*_{a,b} = \int_a^b g(t^*,s)\,\mathrm{d}s$. Then

$$\begin{aligned}\left(B^{-1} - A^{-1}\right)u(t^*) &= \left(B^{-1} - A^{-1}\right)N(u)(t^*)\\ &= A^{-1}B^{-1}F(u)(t^*)\\ &= \int_0^T g(t^*,s)\,B^{-1}F(u)(s)\,\mathrm{d}s\\ &\ge \int_a^b g(t^*,s)\,B^{-1}F(u)(s)\,\mathrm{d}s\\ &\ge \int_a^b g(t^*,s)\,B^{-1}\sigma\mathrm{d}s = g^*_{a,b}B^{-1}\sigma.\end{aligned}$$

This together with (11) implies

$$\left|\left(B^{-1}-A^{-1}\right)u\left(t^{*}\right)\right|_{q}\geq g_{a,b}^{*}\frac{\sigma}{\left|\varphi_{1}\right|_{\infty}\left(\lambda_{1}+m\right)}\left|\varphi_{1}\right|_{q}.$$

Therefore, (7) holds. □

Remark 1. The assumption $\gamma \geq \frac{n-2}{n+2}$ is not essential since γ can always be increased by changing c in (5) correspondingly.

3. NONLINEARITIES WITH SUBCRITICAL SUPERLINEAR GROWTH

For the next result we assume that $f(t,u)$ has a subcritical superlinear growth in u.

Theorem 2. *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be continuous and satisfies* (5) *for some $c,d>0$, $1\leq\gamma<2^{*}-1=\frac{n+2}{n-2}$ if $n\geq 3$ and $1\leq\gamma<\infty$ if $n=1$ or $n=2$. Let $\left|B^{-1}\right|$ denotes the norm of operator B^{-1} from $L^{r}(\Omega)$ to $L^{q}(\Omega)$ where $r=\left(2^{*}\right)'$, $q=2^{*}$ if $n\geq 3$ and $r\geq 1$, $q\geq 1$ in case that $n=1$ or $n=2$. Denote $c^{*}=c\left|1\right|_{r}$. Assume that there exists $R>0$ such that*

$$\left|B^{-1}\right|\left(c^{*}+dR^{\gamma}\right)\leq R. \tag{12}$$

Then problem (1) *has at least one solution u with $|u|_{\infty,p}\leq R$.*

If the additional assumption on f in Theorem 1 *holds, then the solution u satisfies* (7).

Proof. Let $p=\gamma r$. Notice that for $n\geq 3$, from $1\leq\gamma<2^{*}-1=\frac{2^{*}}{\left(2^{*}\right)'}$, we have $\left(2^{*}\right)'\leq p=\gamma\left(2^{*}\right)'<2^{*}$.

Let $F:C\left(\left[0,T\right];L^{p}\left(\Omega\right)\right)\to C\left(\left[0,T\right];L^{r}\left(\Omega;\mathbb{R}_{+}\right)\right)$ be given by

$$F\left(u\right)\left(t\right)=f\left(t,u\left(t\right)\right),\quad u\in C\left(\left[0,T\right];L^{p}\left(\Omega\right)\right),\ t\in\left[0,T\right].$$

From (5) we immediately see that F is well defined from $C\left(\left[0,T\right];L^{p}\left(\Omega\right)\right)$ to $C\left(\left[0,T\right];L^{r}\left(\Omega\right)\right)$ and

$$\left|F\left(u\right)\left(t\right)\right|_{r}\leq c^{*}+d\left|u\left(t\right)\right|_{p}^{\gamma},\quad t\in\left[0,T\right]. \tag{13}$$

Here again $c^{*}=c\left|1\right|_{r}$. As in the proof of the previous theorem, N is completely continuous from $C\left(\left[0,T\right];L^{p}\left(\Omega\right)\right)$ to itself and satisfies (9). Here $\left|\left(A-B\right)^{-1}\right|$ stands for the norm of the operator $\left(A-B\right)^{-1}$ from the space $C\left(\left[0,T\right];L^{r}\left(\Omega\right)\right)$ to $C\left(\left[0,T\right];L^{p}\left(\Omega\right)\right)$. Consequently (10) holds for R given by (12). The rest of the proof is identical to that of Theorem 1. □

In [5] we use a similar technique in order to obtain existence, localization and multiplicity results for the nonlinear wave equation, via Krasnoselskii's compression-expansion fixed point theorem in cones.

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