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ESTIMATES FOR THE SEMIGROUP ASSOCIATED WITH BERNSTEIN OPERATORS

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. Let $(T(t))_{t\geq 0}$ be the semigroup associated with the classical Bernstein operators. We present some quantitative results concerning the behavior of $T(t)$ as $t \to 0$, respectively $t \to \infty$.

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1. INTRODUCTION

Let $b_{nk}(x) := \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$, $n \geq 1$, $0 \leq k \leq n$. The classical Bernstein operators B_n on $C[0,1]$ are defined by

$$B_n f(x) = \sum_{k=0}^{n} b_{nk}(x) f\left(\tfrac{k}{n}\right). \tag{1}$$

A positive contraction C_0-semigroup $(T(t))_{t\geq 0}$ on $C[0,1]$ is associated with (B_n):

$$T(t)f(x) = \lim_{n\to\infty} B_n^{[nt]} f(x), \tag{2}$$

where $t \geq 0$, $f \in C[0,1]$, $x \in [0,1]$, and the convergence is uniform on $[0,1]$.

Concerning (2), see [1], Section 6.3. Other representations of the semigroup are presented in [3] and [4].

It is well-known that for $f \in C[0,1]$,

$$\lim_{t\to 0} T(t)f = f \tag{3}$$

and

$$\lim_{t\to\infty} T(t)f = Tf, \tag{4}$$

where T is the projection defined by

$$Tf(x) = (1-x)f(0) + xf(1). \tag{5}$$

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This paper contains some quantitative results concerning (3) and (4).

2. APPROXIMATING f

Let $e_j(x) = x^j$, $x \in [0,1]$, $j = 0,1,2$. Then

$$B_n e_0 = e_0, \quad B_n e_1 = e_1, \quad B_n e_2 = e_2 + (e_1 - e_2)/n. \tag{6}$$

Theorem 1. *For $f \in C^2[0,1]$, $x \in [0,1]$, and $t \geq 0$,*

$$|T(t)f(x) - f(x)| \leq (1 - \mathrm{e}^{-t})\frac{x(1-x)}{2}\|f''\|. \tag{7}$$

Proof. From (6) we obtain

$$B_n^{[nt]} e_2 = \left(1 - \frac{1}{n}\right)^{[nt]} e_2 + \left(1 - \left(1 - \frac{1}{n}\right)^{[nt]}\right) e_1.$$

Using (2) we get

$$T(t)e_2 = \mathrm{e}^{-t} e_2 + (1 - \mathrm{e}^{-t})e_1. \tag{8}$$

It is easy to see that $T(t)e_0 = e_0$ and $T(t)e_1 = e_1$. This means that for given $t \geq 0$ and $x \in [0,1]$,

$$g \to T(t)g(x), \quad g \in C[0,1],$$

is a probability Radon measure with barycenter x.

Since $\frac{1}{2}\|f''\|e_2 \pm f$ are convex functions, we have

$$\tfrac{1}{2}\|f''\|T(t)e_2(x) \pm T(t)f(x) \geq \tfrac{1}{2}\|f''\|x^2 \pm f(x). \tag{9}$$

Combining (8) and (9) we get (7). □

3. APPROXIMATING Tf

Let $\mathcal{L}$ be the set of all the functions $f \in C[0,1]$ for which there exist constants C_f and K_f such that

$$|f(x) - f(0)| \leq C_f x, \quad x \in [0,1], \tag{10}$$

and

$$|f(1) - f(x)| \leq K_f(1 - x), \quad x \in [0,1]. \tag{11}$$

Theorem 2. (i) *If $f \in \mathcal{L}$, then*

$$|T(t)f(x) - Tf(x)| \leq (C_f + K_f)\mathrm{e}^{-t}x(1-x), \tag{12}$$

for all $t \geq 0$, $x \in [0,1]$.

(ii) *If $f \in C[0,1]$ and there exists a constant M such that*

$$|T(t)f(x) - Tf(x)| \leq M\mathrm{e}^{-t}x(1-x), \quad t \geq 0,\ x \in [0,1], \tag{13}$$

then $f \in \mathcal{L}$.

Proof. (i) Let $f \in \mathcal{L}$. Then

$$\begin{aligned}|B_n f(x) - Tf(x)| =&|B_n(f - Tf)(x)| \\ =&\left|\sum_{k=0}^{n} b_{nk}(x)\left(f\left(\tfrac{k}{n}\right) - \left(1 - \tfrac{k}{n}\right) f(0) - \tfrac{k}{n} f(1)\right)\right| \\ \leq&\sum_{k=0}^{n} b_{nk}(x)\left|\left(1 - \tfrac{k}{n}\right)\left(f\left(\tfrac{k}{n}\right) - f(0)\right) + \tfrac{k}{n}\left(f\left(\tfrac{k}{n}\right) - f(1)\right)\right| \\ \leq&(C_f + K_f)\sum_{k=0}^{n} b_{nk}(x)\left(\tfrac{k}{n} - \left(\tfrac{k}{n}\right)^2\right) \\ =&(C_f + K_f)\left(1 - \tfrac{1}{n}\right) x(1 - x).\end{aligned}$$

We infer that

$$|B_n^{[nt]} f(x) - Tf(x)| \leq (C_f + K_f)\left(1 - \tfrac{1}{n}\right)^{[nt]} x(1 - x),$$

which implies (12).

(ii) Suppose that (13) is satisfied. For $t = 0$ we get

$$|f(x) - (1 - x)f(0) - xf(1)| \leq Mx(1 - x), \quad x \in [0, 1].$$

Now it is easy to infer that $f \in \mathcal{L}$. □

Remark 1. Theorem 3.1(i) was proved with a different method in [2]. □

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