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REMARKS ABOUT A PAPER DEALING WITH THE EQUIVALENCE OF MANN AND ISHIKAWA ITERATIONS*

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Abstract. We give an affirmative answer to the following question: are Mann and Ishikawa iterations equivalent under the assumptions that $\lim_{n\to\infty} \alpha_n \neq 0$ and $\lim_{n\to\infty} \beta_n \neq 0$?

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THE RESULT

Let X be a Banach space. The map $J : X \to 2^{X^*}$ given by

$$Jx := \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\}, \quad \forall x \in X, \tag{1}$$

is called *the normalized duality mapping.* The Hahn-Banach theorem assures that $Jx \neq \emptyset, \forall x \in X$.

Definition 1. *Let X be a real Banach space. Let B be a nonempty subset. A map $T : B \to B$ is called strongly pseudocontractive if there exists $k \in (0,1)$ and a $j(x-y) \in J(x-y)$ such that*

$$\langle Tx - Ty, j(x-y)\rangle \leq k\,\|x-y\|^2, \quad \forall x, y \in B. \tag{2}$$

A map $S : X \to X$ is called strongly accretive if there exists $k \in (0,1)$ and a $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y)\rangle \geq k\,\|x-y\|^2, \quad \forall x, y \in X. \tag{3}$$

Remark 1. The operator T is strongly pseudocontractive map if and only if $(I-T)$ is strongly accretive. □

Let X be a real Banach space, B be a nonempty, convex subset of X, and $T : B \to B$ be an operator. Let $u_1, x_1 \in B$. The following iteration is known as Mann iteration, see [3]:

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n, \tag{4}$$

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where the sequence satisfies $0 \leq \alpha_n \leq 1,\ \forall n \in \mathbb{N}$. Ishikawa iteration is given by, see [1]:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \qquad (5)$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n,$$

where the sequences satisfy $0 \leq \alpha_n, \beta_n \leq 1,\ \forall n \in \mathbb{N}$.

Remark 2. (i) In all papers *(*[2]*)*, [4], [5] and [6], the assumptions on T, B and X are the same, with $B \subset X$ and $T : B \to B$.

(ii) For the above T, B, X, we have shown in [4] that Mann and Ishikawa iterations are equivalent when

$$0 \leq \alpha_n, \beta_n \leq 1,\ \forall n \in \mathbb{N},\ \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty. \qquad (6)$$

(iii) In [5] and [6], the authors have additional assumptions on $\{\alpha_n\}$ and $\{\beta_n\}$, without changing the most important conditions from (6) :

$$0 \leq \alpha_n, \beta_n \leq 1,\ \forall n \in \mathbb{N},\ \sum_{n=0}^{\infty} \alpha_n = \infty. \qquad (7)$$ □

All convergence results from [5] and [6] are dealing with

$$\lim_{n\to\infty} \alpha_n \neq 0 \text{ and } \lim_{n\to\infty} \beta_n \neq 0. \qquad (8)$$

In our Theorem 4 from [4], we have the following condition satisfied, (see (6)):

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0. \qquad (9)$$

It seems that it is impossible to apply Theorem 4 from [4] in the case (8). This is only an appearance.

Theorem 2. *Let B be a closed convex subset of an arbitrary Banach space X and let T be a Lipschitzian strongly pseudocontractive selfmap of B, with Lipschitz constant $L \geq 1$. Let $x_0 = u_0 \in B$ and $\{x_n\}$ and $\{u_n\}$ be the Ishikawa and Mann iterations* (5) *and* (4)*, with $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying* (7) *and*

$$\alpha_n + \beta_n \leq \frac{k(1-k)}{2+(2-k)(1+L^2)+L^2(1+L)}, \forall n \geq n_0. \qquad (10)$$

Then the following are equivalent:

(i) *the Mann iteration converges to x^*,*

(ii) *the Ishikawa iteration converges to x^*.*

Proof. The proof is exactly the same as the proof of Theorem 4 from [4]. The difference now is that we cannot apply formula (30) from page 457, from [4]. However, this formula is satisfied now by the use of (10). Note that (9) can be replaced successfully with (10). □

Let S be a strongly accretive operator. Let us consider when the equation $Sx = f$ has a solution for a given $f \in X$. It easy to see that

$$Tx = x + f - Sx, \quad \forall x \in X, \tag{11}$$

is a strongly pseudocontractive operator. A fixed point for T is the solution of $Sx = f$, and conversely. It is well known that if S is bounded $(I - S)$ could be unbounded, for example take $S : \mathbb{R} \to B := [-1, 1]$ with $S(x) = (1/2)\cos x$. Remark 1, Definition 1 and Theorem 2 lead us to the following result.

THEOREM 3. *Let X be an arbitrary Banach space and let T be a Lipschitzian strongly accretive selfmap of X, with $S(X)$ bounded and Lipschitz constant $L \geq 1$. Let $x_0 = u_0 \in B$ and $\{x_n\}$ and $\{u_n\}$ be the Ishikawa and Mann iterations* (5) *and* (4), *with $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying* (7) *and* (10). *Then the following are equivalent:*

(i) *the Mann iteration with $Tx = f + (I - S)x$ converges to x^**

(ii) *the Ishikawa iteration with $Tx = f + (I - S)x$ converges to x^*.*

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