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LOCAL-GLOBAL EFFICIENCY PROPERTIES FOR MULTIOBJECTIVE MAX-MIN PROGRAMMING

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Abstract. The purpose of this paper is to give sufficient conditions of generalized concavity and convexity type for a local (weakly) max-min efficient solution to be a global (weakly) max-min efficient solution for an vector maxmin programming problem.

In the particular case of the vector max-min pseudomonotonic programming problem, we derive some characterizations properties of max-min efficient and properly max-min efficient solutions .

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1. INTRODUCTION

The aim of this paper is to derive sufficient conditions of generalized concavity type for a local (weakly) max-min efficient solution to be a global (weakly) max-min efficient solution for an vector max-min set-valued programming problem.

Let $X \subset \mathbb{R}^n$ $Y \subset \mathbb{R}^m$and $Q : X \times Y \longrightarrow \mathbb{R}^p$ be a vectorial function defined on $X \times Y$.

The multiobjective max-min programming problem under consideration is formulated as

$$\text{(VMMP.)} \qquad \text{Vmax-min } Q(x,y), \quad \text{subject to } x \in X, y \in Y,$$

where the vector maximin "Vmax-min" will be understood in the sense of efficiency that will be defined in different forms below in the next section.

The optimal solutions of the $VMMP$ that we deal with include the concepts of efficient, weakly efficient, local efficient and properly efficient solutions that will be defined with respect to a semiorder relationship in $\mathbb{R}^p$.

The paper is organized as follows. In Section 2 we introduce the notation and definitions, which will be used throughout of the paper.

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In Section 3, we give sufficient conditions of generalized concavity type for a local (weakly) efficient solution to be a global (weakly) efficient solution of a $VMMP$.

In Section 4, for the particular case of the vector max-min pseudomonotonic programming problem, we obtain some characterizations properties of efficient and properly efficient solutions.

Some concluding remarks are made in the last section.

2. NOTATION AND DEFINITIONS

Next we recall some notations and concepts considered in [6] and introduce some max-min efficiency concepts.

Let $a, b \in \mathbb{R}^p$ be arbitrary vectors in $\mathbb{R}^p$. Then we consider the following relations on the set $\mathbb{R}^p$:

(i) $a \geq b$ if and only if $a_i \geq b_i$, for any $i \in J = \{1, 2, ..., p\}$,

(ii) $a > b$ if and only if $a_i \geq b_i$, for any $i \in J$ and there is $j \in J$ such that $a_j > b_j$,

(iii) $a \gg b$ if and only if $a_i > b_i$, for any $i \in J$.

Next, we consider some classes of generalized concave functions (see, e.g. [4], [2], [3], [6]).

Definition 1. *Let $F : X \longrightarrow \mathbb{R}^p$ be a vector function, where X is a convex non-empty set in $\mathbb{R}^s$. We say that F is:*

a) quasiconcave *if for any $x', x'' \in X$ and $t \in (0, 1)$, we have $F(tx' + (1 - t)x'') \geq Min(F(x'), F(x''))$,*

b) semistrictly quasiconcave *if for any $x', x'' \in X$ such that $F(x') \neq F(x'')$, we have $F(tx' + (1 - t)x'') > Min(F(x'), F(x''))$, for each $t \in (0, 1)$,*

c) semiexplicitly quasiconcave *if it is quasiconcave and semistrictly quasiconcave,*

d) strictly quasiconcave *if for all $x', x'' \in X$ such that $F(x') \neq F(x'')$, we have $F(tx' + (1 - t)x'') \gg Min(F(x'), F(x''))$, for each $t \in (0, 1)$,*

e) explicitly quasiconcave *if it is quasiconcave and strictly quasiconcave,*

f) quasiconvexe, semi strictly quasiconvex, semiexplicitly quasiconvex, strictly quasiconvex *and* explicitly quasiconvex *if $(-F)$ is quasiconcave, semistrictly quasiconcave, semiexplicitly quasiconcave, strictly quasiconcave and explicitly quasiconcave respectively.*

Obviously, from Definition 1, F is semiexplicitly quasiconcave if it is explicitly quasiconcave. However, the converse is not true (see, e.g. [2]).

Definition 2. *Let D be an open convex set $D \subseteq \mathbf{R}^n$.*

i) *A differentiable function $f : D \to \mathbf{R}$ is pseudoconvex if*

$$\nabla f(x)(y - x) \geq 0 \Longrightarrow f(y) \geq f(x), \forall x, y \in D,$$

where $\nabla f(x)$ is the gradient of the function f on the point x.

ii) *The differentiable function* $f : D \to \mathbf{R}$ *is called pseudomonotonic if* f *and* $-f$ *are pseudoconvex.*

Next we consider for Problem $MSVP$ some efficiency concepts based on the semiorder relationships presented above (see, (i)–(iii)).

Definition 3. *A point* $(\overline{x}, \overline{y}) \in X \times Y$ *is said to be a* max-min efficient solution *to Problem* $VMMP$ *if there does not exist* $x \in X$ *such that* $Q(x, \overline{y}) > Q(\overline{x}, \overline{y})$ *(that is,* $\overline{x}$ *is a maximum-efficient solution for* $Q(\cdot, \overline{y})$ *on* X*) and there does not exist* $y \in Y$ *such that* $Q(\overline{x}, \overline{y}) > Q(\overline{x}, y)$ *(that is,* $\overline{y}$ *is a minimum-efficient solution for* $Q(\overline{x}, \cdot)$ *on* Y*).*

Let MmE denote the set of all efficient max-min solutions to Problem $VMMP$.

Definition 4. *A point* $(\overline{x}, \overline{y}) \in X \times Y$ *is said to be a* weakly max-min efficient solution *to Problem* $VMMP$ *if there does not exist* $x \in X$ *such that* $Q(x, \overline{y}) \gg Q(\overline{x}, \overline{y})$ *(that is* $\overline{x}$ *is a weakly maximum-efficient solution for* $Q(\cdot, \overline{y})$ *on* X*) and there does not exist* $y \in Y$ *such that* $Q(\overline{x}, \overline{y}) \gg Q(\overline{x}, y)$ *(that is,* $\overline{y}$ *is a weakly minimum-efficient solution for* $Q(\overline{x}, \cdot)$ *on* Y*).*

Let $MmWE$ denote the set of all max-min efficient solutions to Problem $VMMP$.

Definition 5. i) *A point* $(\overline{x}, \overline{y}) \in X \times Y$ *is said to be a* local max-min efficient solution *to Problem* $VMMP$ *if there does not exist* $x \in X \cap U$ *such that* $Q(x, \overline{y}) > Q(\overline{x}, \overline{y})$ *(that is* $\overline{x}$ *is a local maximum-efficient solution for* $Q(\cdot, \overline{y})$ *on* X*) and there does not exist* $y \in Y \cap V$ *such that* $Q(\overline{x}, \overline{y}) > Q(\overline{x}, y)$ *(that is* $\overline{y}$ *is a local minimum-efficient solution for* $Q(\overline{x}, \cdot)$ *on* Y*), for some neighborhoods* U *of* $\overline{x}$ *and* V *of* $\overline{y}$*.*

ii) *A point* $(\overline{x}, \overline{y}) \in X \times Y$ *is said to be a* local weakly max-min efficient solution *to Problem* $VMMP$ *if there does not exist* $x \in X \cap U$ *such that* $Q(x, \overline{y}) \gg Q(\overline{x}, \overline{y})$ *(that is* $\overline{x}$ *is a local weakly maximum-efficient solution for* $Q(\cdot, \overline{y})$ *on* X*) and there does not exist* $y \in Y \cap V$ *such that* $Q(\overline{x}, \overline{y}) \gg Q(\overline{x}, y)$ *(that is,* $\overline{y}$ *is a local weakly minimum-efficient solution for* $Q(\overline{x}, \cdot)$ *on* Y*), for some neighborhoods* U *of* $\overline{x}$ *and* V *of* $\overline{y}$*.*

Let $MmLE$ ($MmLWE$) denote the set of all local (weakly) max-min efficient solutions to Problem $VMMP$.

Definition 6. *Let* $Q(x, y) = (Q_1(x, y), ..., Q_p(x, y))$*, for any* $(x, y) \in X \times Y$*. A max-min efficient solution* $(\overline{x}, \overline{y}) \in X \times Y$ *to Problem* $VMMP$ *is said to be a* properly max-min efficient solution *if there exists a scalar* $M > 0$ *such that:*

i) *for all* $i \in J$ *and each* $x \in X$*, for which* $Q_i(x, \overline{y}) > Q_i(\overline{x}, \overline{y})$*, there exists* $j \in J - \{i\}$*, for which* $Q_j(x, \overline{y}) > Q_j(\overline{x}, \overline{y})$ *and* $\frac{Q_i(x,\overline{y}) - Q_i(\overline{x},\overline{y})}{Q_j(\overline{x},\overline{y}) - Q_j(x,\overline{y})} \leq M$ *(that is,* $\overline{x}$ *is a properly maximum-efficient solution for* $Q(\cdot, \overline{y})$ *on* X*);*

ii) *for all* $i \in J$ *and each* $y \in Y$, *for which* $Q_i(\overline{x}, y) < Q_i(\overline{x}, \overline{y})$, *there exists* $j \in J - \{i\}$, *for which* $Q_j(\overline{x}, y) > Q_j(\overline{x}, \overline{y})$ *and* $\frac{Q_i(\overline{x},y)-Q_i(\overline{x},\overline{y})}{Q_j(\overline{x},\overline{y})-Q_j(\overline{x},y)} \leq M$ *(that is,* $\overline{y}$ *is a properly minimum-efficient solution for* $Q(\overline{x}, \cdot)$ *on* Y*).*

Let $MmPE$ denote the set of all properly efficient solutions to Problem $VMMP$.

From Definition 6, it follows that $(\overline{x}, \overline{y}) \in X \times Y$ is a *properly max-min efficient solution* to Problem $VMMP$ if and only if $\overline{x}$ is a properly maximum-efficient solution for $Q(\cdot, \overline{y})$ on X and $\overline{y}$ is a properly minimum-efficient solution for $Q(\overline{x}, \cdot)$ on Y.

Obviously, from Definitions 3–6, we have the following relationship between the different classes of optimal solutions of $VMMP$:

$$MmPE \subset MmE \subset MmWE, \tag{1}$$

$$MmE \subset MmLE, \tag{2}$$

$$MmWE \subset MmLWE, \tag{3}$$

$$MmLE \subset MmLWE. \tag{4}$$

The efficiency notions given by Definitions 3-5 are analogous to that considered for vector real valued objective functions in refs. [2], [3]. The proper efficiency concept is a generalization of that introduced by Geoffrion [1].

3. SUFFICIENT CONDITIONS FOR LOCAL-GLOBAL EFFICIENCY PROPERTIES

We now generalize to vector max-min optimization problems a characterization of local efficient solutions obtained in Refs. [2], [3] for usual vector optimization problems or for vector set-valued optimization problems [6]. A similar result is given for local weakly efficient solutions.

We mention that some local-global efficiency properties for minimum-risk problems was recently obtained by Tigan and Stancu-Minasian [5].

Given $p \geq 2$ and $X \neq \emptyset$, $X \subseteq \mathbb{R}^n$an arbitrary set, consider the function $f : X \to \mathbb{R}^p$, $f(x) = (f_1(x), f_2(x), \ldots, f_p(x))$, $\forall x \in X$, where $f_i : X \to \mathbb{R}$, for any $i \in J = \{1, 2, \ldots, p\}$.

Next, we need the following property due to Weber [7] which provide sufficient conditions in order to a pseudo-monotonic multi-objective program has "complete proper efficiency property", that is any efficient solution to this problem is also properly efficient.

LEMMA 7. [7] *Let* f_k *(*$k \in \{1, 2, \ldots, p\}$ *be pseudo-monotonic functions twice continuously differentiable on the open subset* $D \subseteq \mathbf{R}^n$ *and* X *be a polyhedral set* $X \subseteq D$. *Then a point* $x' \in X$ *is maximum (minimum) efficient for the pseudo-monotonic function* $f = (f_1, f_2, \ldots, f_p)$ *on* X *if and only if it is properly maximum (minimum) efficient.*

THEOREM 8. *Let* $X \subset \mathbb{R}^n$ *and* $Y \subset \mathbb{R}^m$ *be given non-empty convex sets and* $Q : X \times Y \longrightarrow \mathbb{R}^p$ *be a vector function such that: (i)* $Q(\cdot, y) : X \longrightarrow$

$\mathbb{R}^p$ is semiexplicitly quasiconcave for any $y \in Y$; (ii) $Q(x, \cdot) : Y \longrightarrow \mathbb{R}^p$ is semiexplicitly quasiconvex for any $x \in X$. Then $(\overline{x}, \overline{y}) \in X \times Y$ is a local max-min efficient solution to Problem $VMMP$ if and only if $(\overline{x}, \overline{y})$ is a (global) max-min efficient solution (i.e. $MmE = MmLE$).

Proof. From (2) we have $MmE \subset MmLE$. To prove the converse inclusion, assume to the contrary that $(\overline{x}, \overline{y}) \in X \times Y$ is a local max-min efficient solution (with respect to the neighborhoods U of $\overline{x}$ and V of $\overline{y}$), which is not a global max-min efficient solution. This means that: (a) there exists $x^0 \in X$ such that $Q(x^0, \overline{y}) > Q(\overline{x}, \overline{y})$ or (b) there exists $y^0 \in Y$ such that $Q(\overline{x}, y^0) < Q(\overline{x}, \overline{y})$. In the case (a), since $Q(\cdot, \overline{y})$ is semiexplicitly quasiconcave, we have

$$Q(tx^0 + (1-t)\overline{x}, \overline{y}) > Q(\overline{x}, \overline{y}), \text{ for all } t \in (0,1). \tag{5}$$

But for t sufficiently close to zero, $x(t) = tx^0 + (1-t)\overline{x}$ will be in the neighborhood U of $\overline{x}$. But this shows by (5) and Definition 3 that $(\overline{x}, \overline{y})$ would not be a local max-min efficient solution, which is a contradiction.

In the case (b), since $Q(\overline{x}, \cdot)$ is semiexplicitly quasiconvex, we have

$$Q(\overline{x}, ty^0 + (1-t)\overline{y}) < Q(\overline{x}, \overline{y}), \text{ for all } t \in (0,1). \tag{6}$$

But for t sufficiently close to zero, $y(t) = ty^0 + (1-t)\overline{y}$ will be in the neighborhood V of $\overline{y}$. But this shows by (6) and Definition 3 that $(\overline{x}, \overline{y})$ would not be a local max-min efficient solution, which is a contradiction too.

Therefore, we proved that $MmLE \subset MmE$, which toogether with (2) implies the equality $MmE = MmLE$. □

We supplement the result in Theorem 8 by a similar one for weakly efficient solutions.

THEOREM 9. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be given non-empty convex sets and $Q : X \times Y \longrightarrow \mathbb{R}^p$ be a vector function such that:*

(i) *$Q(\cdot, y) : X \longrightarrow \mathbb{R}^p$ is explicitly quasiconcave for any $y \in Y$;*

(ii) *$Q(x, \cdot) : Y \longrightarrow \mathbb{R}^p$ is explicitly quasiconvex for any $x \in X$. Then $(\overline{x}, \overline{y}) \in X \times Y$ is a local weakly max-min efficient solution to Problem $VMMP$ if and only if $(\overline{x}, \overline{y})$ is a (global) weakly max-min efficient solution.*

Proof. One can follow the lines of previous proof. To prove the nontrivial implication, assume to the contrary that $(\overline{x}, \overline{y}) \in X \times Y$ is a local weakly max-min efficient solution (with respect to the neighborhoods U of $\overline{x}$ and V of $\overline{y}$), which is not a global weakly max-min efficient solution. This means that: (a) there exists $x^0 \in X$ such that $Q(x^0, \overline{y}) \gg Q(\overline{x}, \overline{y})$ or (b) there exists $y^0 \in Y$ such that $Q(\overline{x}, y^0) \ll Q(\overline{x}, \overline{y})$. In the case (a), since $Q(\cdot, \overline{y})$ is semiexplicitly quasiconcave, we have

$$Q(tx^0 + (1-t)\overline{x}, \overline{y}) \gg Q(\overline{x}, \overline{y}), \text{ for all } t \in (0,1). \tag{7}$$

But for t sufficiently close to zero, $x(t) = tx^0 + (1-t)\overline{x}$ will be in the neighborhood U of $\overline{x}$. But this shows by (7) and Definition 4 that $(\overline{x}, \overline{y})$ would not be a local max-min efficient solution, which is a contradiction.

In the case (b), since $Q(\overline{x}, \cdot)$ is semiexplicitly quasiconvex, we have

$$Q(\overline{x}, ty^0 + (1-t)\overline{y}) \ll Q(\overline{x}, \overline{y}), \text{ for all } t \in (0,1). \tag{8}$$

But for t sufficiently close to zero, $y(t) = ty^0 + (1-t)\overline{y}$ will be in the neighborhood V of $\overline{y}$. But this shows by (8) and Definition 4 that $(\overline{x}, \overline{y})$ would not be a local max-min efficient solution, which is a contradiction too. □

Theorem 10. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be given non-empty convex sets and $Q : X \times Y \longrightarrow \mathbb{R}^p$ be a vector function such that:*

(i) *$Q_i(\cdot, y) : X \longrightarrow \mathbb{R}$ is pseudomonotonic twice continuously differentiable on the open subset $D \subseteq \mathbb{R}^n$ for any $y \in Y$ and $i = 1, 2, ..., p$, where $X \subset D$;*

(ii) *$Q_i(x, \cdot) : Y \longrightarrow \mathbb{R}$ is pseudomonotonic twice continuously differentiable on the open subset $E \subseteq \mathbb{R}^m$ for any $x \in X$ and $i = 1, 2, ..., p$, where $Y \subset E$.*

Then $(\overline{x}, \overline{y}) \in X \times Y$ is a max-min efficient solution to Problem $VMMP$ if and only if $(\overline{x}, \overline{y})$ is a properly max-min efficient solution.

Proof. From (1) we have $MmPE \subset MmE$. In order to prove the converse inclusion, let $(\overline{x}, \overline{y}) \in X \times Y$ be a max-min efficient solution to Problem $VMMP$. Then, by Definition 3, it follows that $\overline{x}$ is a maximum-efficient solution for $Q(\cdot, \overline{y})$ on X and $\overline{y}$ is a minimum-efficient solution for $Q(\overline{x}, \cdot)$ on Y. From hypothesis (i) and (ii), by Lemma 7, it follows that $\overline{x}$ is a properly maximum-efficient solution for $Q(\cdot, \overline{y})$ on X and $\overline{y}$ is a properly minimum-efficient solution for $Q(\overline{x}, \cdot)$ on Y. Then, by Definition 6, it result that $(\overline{x}, \overline{y})$ is a properly max-min efficient solution to Problem $VMMP$. □

4. CONCLUSIONS

In this paper we obtained sufficient conditions implying generalized concavity and convexity assumptions of the objective functions, in order to a local (weakly) efficient solution be a global (weakly) efficient solution for an vector max-min programming problem.

In the particular case of the vector pseudomonotonic max-min programming problem, we derived some characterizations properties of efficient and properly efficient solutions.

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