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ON β-DIFFERENTIABILITY OF NORMS*

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. In this note we give some characterizations for the differentiability with respect to a bornology of a continuous convex function. The special case of seminorms is treated. A characterization of this type of differentiability in terms of the subgradient of the function is also obtained.

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RESULTS

Let E and E_1 be Banach spaces, U an open subset in E and $x \in U$. A function $f : U \to E_1$ is said to be *Gâteaux differentiable* at x if there exists a linear continuous mapping denoted $\mathrm{d}f(x) : E \to E_1$ such that for each h in E one has

$$\mathrm{d}f(x)(h) = \lim_{t\to 0+} \tfrac{1}{t}(f(x+th) - f(x)). \tag{1}$$

The function f is said to be *Fréchet differentiable* at x if there exists a linear continuous mapping denoted $f'(x) : E \to E_1$ such that for each $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$\|f(x+h) - f(x) - f'(x)(h)\| \leq \varepsilon\|h\|, \quad \text{for each } h \in B_E(x,\delta). \tag{2}$$

The two linear mappings $df(x)$, $f'(x)$ are the *Gâteaux* and *Fréchet differentials* and are unique (when they exist).

In the sequel, we shall be interested only by real functions (i.e. $E_1 = \mathbb{R}$). When a real function f is also convex on an open convex set $U \subseteq E$, then the limit in (1) exists and is denoted by $\mathrm{d}^+f(x)$; this directional derivative is generally only sublinear and the Gâteaux differentiability of f at x is equivalent with the linearity of $\mathrm{d}^+f(x)$, or with the fact that

$$\mathrm{d}^+f(x)(h) = -\mathrm{d}^+f(x)(-h) \ [=: \mathrm{d}^-f(x)(h)], \quad \text{for each } h \in E.$$

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It is obvious that any Fréchet differentiable function is also Gâteaux differentiable, and the two differentials coincide. The converse is not true, even for convex functions. For example, the norm of the Banach space ℓ^1 is known to be nowhere Fréchet differentiable, but it is Gâteaux differentiable at those points $(x_n)_{n\in\mathbb{N}}$ having only nonzero components.

It is well known that the function f is Fréchet differentiable at x if and only if it is Gâteaux differentiable at x and the limit (1) is uniform with respect to $h \in B[0,1]$ (=the closed unit ball in E) or, equivalently, with respect to any bounded subset of E. This remark allows a useful generalization of the differentiability.

Let β be a nonempty family of bounded sets in E whose union is E, which is directed with respect to $\subseteq$ (i.e., for each B_1, $B_2 \in \beta$ there exists $B_3 \in \beta$ such that B_1, $B_2 \subseteq B_3$) and is invariant under scalar multiplication. Such a family is named *bornology* in Phelps' monograph [3].

The function f is said to be *β-differentiable* at the point x if f is Gâteaux differentiable at x and the limit (1) is uniform in $h \in B$ for each $B \in \beta$. This turns out to be equivalent with the convergence in the uniform structure $F_\beta(E,\mathbb{R})$. We shall denote by τ_β the topology induced by this uniform structure.

The following interesting special cases of a bornology arise (as pointed out in [3]):

- $\beta = G =$ the family of all finite subsets in E (generating the Gâteaux differentiability);
- $\beta = F =$ the family of all bounded subsets in E (generating the Fréchet differentiability);
- $\beta = H =$ the family of all compact subsets in E (generating the Hadamard differentiability);
- $\beta = W =$ the family of all weak compact subsets in E (generating the strong Hadamard differentiability).

One obviously has the inclusions: $G \subseteq \beta \subseteq F$, $G \subseteq H \subseteq W \subseteq F$; if f is β_2-differentiable and $\beta_1 \subseteq \beta_2$, then f is also β_1-differentiable and the two differentials coincide.

THEOREM 1. *Let f be a continuous convex function on an open convex subset U in the normed space E and β a bornology on E. Then f is β-differentiable at $x \in U$ if and only if, for each $B \in \beta$, the limit*

$$\lim_{t\to 0+} \tfrac{1}{t}\left(f(x+th) + f(x-th) - 2f(x)\right) = 0, \tag{3}$$

holds uniformly for $h \in B$.

Proof. Necessity. Let B be an arbitrary subset in β. Using (1) for B and $-B$ one obtains the equalities $\mathrm{d}f(x)(h) = \lim_{t\to 0+} \frac{1}{t}(f(x+th) - f(x))$ and $\mathrm{d}f(x)(-h) = \lim_{t\to 0+} \frac{1}{t}(f(x-th) - f(x))$, which hold uniformly for $h \in B$; by addition the desired conclusion follows.

Sufficiency. Choose $B \in \beta$, $\varepsilon > 0$. Using the continuity of f, one can select a subgradient $x^* \in \partial f(x)$. The hypothesis guarantees the existence of a positive number δ such that $f(x+th) + f(x-th) - 2f(x) < t\varepsilon$, for each $h \in B$ and $t \in (0, \delta)$. (B is bounded, so, for sufficiently small $\delta > 0$ one has $x \pm th \in B$.)

We have

$$\langle x^*, th \rangle \leq f(x+th) - f(x),$$
$$\langle x^*, -th \rangle \leq f(x-th) - f(x),$$

and for $0 < t < \delta$, $h \in B$ one obtains:

$$\begin{aligned} 0 \leq & f(x+th) - f(x) - \langle x^*, th \rangle \\ = & \big(f(x+th) + f(x-th) - 2f(x)\big) + \big(f(x) - f(x-th) - \langle x^*, th \rangle\big) \\ \leq & \varepsilon t + 0 = \varepsilon t, \end{aligned}$$

which implies that (1) holds uniformly for $h \in B$. □

REMARK 1. **a)** For the Fréchet differentiability it is sufficient that the limit (3) holds uniformly on the unit sphere S_E; for the Gâteaux differentiability, the pointwise limit in (3) suffices.

b) The continuity condition imposed on the convex function f cannot be omitted when E is infinite dimensional; in fact it is sufficient to consider a linear discontinuous functional f (cf. [5, p. 251]); in this case, $\mathrm{d}f(x) = f$ is not continuous. □

COROLLARY 2. *Let f_n $(n \in \mathbb{N})$ be a sequence of continuous convex functions on an open convex subset in a Banach space E endowed with a bornology β. If the series $\sum_{n\in\mathbb{N}} f_n$ is pointwise convergent having a continuous sum f, and f is β-differentiable at a point x_0, then each function f_n is β-differentiable at x_0.*

Proof. The statement follows immediately from the preceding theorem, using the relations:

$$\begin{aligned} 0 &\leq \sum_{n\in\mathbb{N}} \tfrac{1}{t}\big(f_n(x+th) + f_n(x-th) - 2f_n(x)\big) \\ &= \tfrac{1}{t}\big(f(x+th) + f(x-th) - 2f(x)\big) \end{aligned}$$

(valid for $x \in U$, $t > 0$, $h \in B \in \beta$ provided that $x + th \in U$). □

The (semi)norms are important special cases of convex functions. The next result represents a simple characterizations for the β-differentiability of a norm, extending a theorem of Smulian [1].

THEOREM 3. *Let E be a normed space endowed with a bornology β and x a point on the unit sphere S_E of E.*

The norm $\|\cdot\|$ *is* β*-differentiable at* x *if and only if the following condition holds: for all sequences* x_n^*, $y_n^* \in S_{E^*}$ *satisfying* $x_n^*(x) \to 1$, $y_n^*(x) \to 1$ *one has* $x_n^* - y_n^* \to 0$ *in* $F_\beta(E, \mathbb{R})$.

Proof. Necessity. Let $B \in \beta$, $\varepsilon > 0$, $x_n^*, y_n^* \in S_{E^*}$ satisfy $x_n^*(x) \to 1$, $y_n^*(x) \to 1$. Choosing $B' \in \beta$ such that $B \cup (-B) \subseteq B'$ and applying the preceding theorem, there exists $\delta > 0$ such that for $t \in (0, \delta]$ one has

$$\|x + th\| + \|x - th\| < 2 + \varepsilon t \leq 2 + \varepsilon\delta, \quad \text{for each } h \in B'.$$

The hypothesis implies the existence of a positive integer n_0 such that for $n \geq n_0$:

$$|1 - x_n^*(x)| + |1 - y_n^*(x)| < \varepsilon\delta.$$

We have

$$x_n^*(x + th) + y_n^*(x - th) \leq \|x + th\| + \|x - th\| \leq 2 + \varepsilon\delta,$$

hence

$$x_n^*(th) - y_n^*(th) \leq 1 - x_n^*(x) + 1 - y_n^*(x) + \varepsilon\delta < 2\varepsilon\delta, \quad \text{for } n \geq n_0.$$

By taking $t = \delta$, one obtains $x_n^*(h) - y_n^*(h) < 2\varepsilon$, for $n \geq n_0$, $h \in B'$.

For $h \in B$, we have $\pm h \in B'$, and the last inequality implies

$$|x_n^*(h) - y_n^*(h)| < 2\varepsilon, \quad \text{for each } n \geq n_0,\ h \in B.$$

Sufficiency. Suppose by contradiction that $\|\cdot\|$ is not β-differentiable at x, and hence there exists $\varepsilon > 0$, $B \in \beta$, $h_n \in B \backslash \{0\}$, $t_n > 0$ such that $t_n \to 0$, $\|x + t_n h_n\| + \|x - t_n h_n\| \geq 2 + \varepsilon t_n$.

Choosing x_n^*, $y_n^* \in S_{E^*}$ such that $x_n^*(x + t_n h_n) \geq \|x + t_n h_n\| - \|t_n h_n\|/n$ and $y_n^*(x - t_n h_n) \geq \|x - t_n h_n\| - \|t_n h_n\|/n$ one obtains

$$\begin{aligned} 1 &\geq x_n^*(x) \\ &= x_n^*(x + t_n h_n) - x_n^*(t_n h_n) \\ &\geq \|x + t_n h_n\| - \tfrac{\|t_n h_n\|}{n} - \|t_n h_n\| \\ &\geq 1 - \tfrac{\|t_n h_n\|}{n} - 2\|t_n h_n\|, \end{aligned}$$

hence $x_n^*(x) \to 1$.

Similarly, $y_n^*(x) \to 1$.

Because

$$x_n^*(x + t_n h_n) + y_n^*(x - t_n h_n) \geq 2 + \varepsilon t_n - 2\tfrac{\|t_n h_n\|}{n},$$

we have

$$x_n^*(h_n) - y_n^*(h_n) \geq \varepsilon - 2\tfrac{\|h_n\|}{n} \geq \tfrac{\varepsilon}{2}, \quad \text{for } n \geq n_0,$$

in contradiction with $x_n^* - y_n^* \to 0$ in $F_\beta(E, \mathbb{R})$ □

If f is a continuous convex function on an open convex subset U of a Banach space E, then the subdifferential ∂f is a set-valued operator, having convex, nonempty weak* compact values in E^*.

We shall obtain a characterization of the β-differentiability for f in terms of the subdifferential operator ∂f. Such characterizations are known for the Fréchet and Gâteaux differentiability, and are very useful in the analysis of the smoothness of f; in the same time such results motivated an intensive research on the set-valued operators.

If (X, τ_1), (Y, τ_2) are topological spaces, a set-valued operator $T : X \to 2^Y$ is said to be τ_1-τ_2 *upper semicontinuous (u.s.c.)* at $x \in X$, if for each subset $W \in \tau_2$ containing $T(x)$, there exists $V \in \tau_1$ containing x such that $T(V) = \cup\{T(v) : v \in V\} \subseteq W$.

The set $\mathrm{dom}(T) := \{x \in X : T(x) \neq \emptyset\}$ is the *domain of* T.

We are interested in the case when the operator T acts between E and 2^{E^*}, where E is a Banach space. Denoting by $\|\cdot\|$ the norm in E and by $\|\cdot\|^*$ its dual norm in E^* we shall consider the strong topology $\tau_{\|\cdot\|}$ (generated by the norm) on E, and the topology τ_β of the β-convergence on E^*, where β is a bornology on E. We remind that $\tau_F = \tau_{\|\cdot\|^*}$, where F is the Fréchet bornology (of all bounded subsets), and τ_G is the weak* topology (G denoting the Gâteaux bornology).

Proposition 4. *Let E be a normed space, β a bornology on E, x a point in E and $T : E \to 2^{E^*}$ a set-valued operator. Then, the following statements are equivalent:*

(i) *T is $\tau_{\|\cdot\|}$-τ_β upper semicontinuous at x.*

(ii) *For each $W \in \tau_\beta$, $T(x) \subseteq W$, $(x_n)_{n\in\mathbb{N}} \subseteq E$ with $\|x_n - x\| \to 0$, there exists $n_0 \in \mathbb{N}$, such that for $n \geq n_0$, $T(x_n) \subseteq W$.*

(iii) *For each $W \in \tau_\beta$, $T(x) \subseteq W$, there exists $\delta > 0$, such that for $r \in (0, \delta]$, $T(B[x, r]) \subseteq W$.*

If, furthermore, $T(x)$ is a singleton $\{x_0^*\}$, the above conditions are also equivalent with

(iv) *For each $(x_n)_{n\in\mathbb{N}} \subseteq E$, with $\|x_n - x\| \to 0$ it follows that*

$$\lim_{n\to\infty} \sup\{|\langle x^* - x_0^*, h\rangle| : x^* \in T(x_n), h \in B\} = 0, \quad (B \in \beta) \tag{4}$$

and, for $\beta = F$, (4) *may be reformulated as*

$$\lim_{r\to 0+} \operatorname{diam} T(B[x, r]) = 0. \tag{5}$$

Proof. The proof is standard, similar to Heine's theorem in general topology. □

We shall need the following result (see [3], [2]):

Theorem 5. *Let E be a normed space, f a convex continuous function defined on an open convex set $D \subseteq E$. Then the subdifferential $\partial f : D \to 2^{E^*}$ is a $\tau_{\|\cdot\|}$-τ_G upper semicontinuous operator.*

Note that in the general case, ∂f will not be $\tau_{\|\cdot\|}$-τ_β upper-semicontinuous for an arbitrary bornology β as the following example shows:

Example 1. *Let $E = \ell^1$ be the Banach space of all summable sequences endowed with the norm $\|x\| = \sum_{n\in\mathbb{N}} |x(n)|$, and $f : E \to \mathbb{R}$, $f(x) = \|x\|$. For $h \in E$, we have*

$$\begin{aligned} d^+ f(x)(h) &= \lim_{t\to 0+} \sum_{n\in\mathbb{N}} \tfrac{|x(n)+th(n)|-|x(n)|}{t} \\ &= \sum_{n\in\mathbb{N}} \lim_{t\to 0} \tfrac{|x(n)+th(n)|-|x(n)|}{t} \\ &= \sum_{n\in\mathbb{N}, x(n)\neq 0} (\operatorname{sign} x(n))h(n) + \sum_{n\in\mathbb{N}, x(n)=0} |h(n)| \end{aligned}$$

(the permutation of the limit and sum symbols can be legitimated by using the Weiersrass theorem, or the dominated convergence theorem from measure theory applied to the sum as a discrete integral). The function f is G-differentiable at x if and only if $\mathrm{d}^+ f(x)(h) = -\mathrm{d}^+ f(x)(-h)$, $(h \in E)$, which means:

$$\sum_{n\in\mathbb{N}, x(n)=0} |h(n)| = 0, \quad \textit{for each } h \in E, \textit{i.e.}, \ x(n) \neq 0, (n \in \mathbb{N}).$$

Choose now $x \in \ell^1$, $x(n) = \alpha_n > 0$ $(n \in \mathbb{N})$; then f is G-differentiable at x.

Defining $x_p = (\alpha_1, \alpha_2, ..., \alpha_p, 0, 0...)$, i.e. $x_p(n) = \alpha_n$ for $n \leq p$ and $x_p(n) = 0$ for $n > p$, we obviously have $\|x_p - x\| \to 0$.

But $\mathrm{d}^+ f(x_p)(h) = h(1) + ... + h(p) + |h(p+1)| + |h(p+2)| + ...$, and by taking $x_p^*(h) = h(1) + ... + h(p)$, one obtains:

$$x_p^* \in E^*, \quad x_p^* \leq \mathrm{d}^+ f(x_p),$$

hence $x_p^* \in \partial f(x_p)$.

On the other hand, $\|\mathrm{d}f(x) - x_p^*\|^* = 1$, so ∂f is not $\tau_{\|\cdot\|}$-τ_F u.s.c. at x (cf. (iii), with $W = B(\mathrm{d}f(x), 1)$).

In this example, f is not F-differentiable at x. This fact will follow from the next theorem which contains also a refinement of the preceding proposition

Theorem 6. *Let E be a normed space, β a bornology, f a continuous convex function on an open convex set $D \subseteq E$, which is β-differentiable at $x \in D$. Then the subdifferential $\partial f : D \to 2^{E^*}$ is an $\tau_{\|\cdot\|}$-τ_β upper semicontinuous operator.*

Proof. Suppose by contradiction that ∂f is not $\tau_{\|\cdot\|}$-τ_β u.s.c. at x. Applying (iv) from Proposition 4 one obtains that there exist $x_n \in E$, with $\|x_n - x\| \to 0$, $\varepsilon > 0$, $B \in \beta$, $h_n \in B$, $x_n^* \in \partial f(x_n)$, such that

$$|\langle x_n^* - x_0^*, h_n\rangle| > 2\varepsilon, \quad (n \in \mathbb{N}), \quad \text{where } x_0^* = \mathrm{d}f(x).$$

Chose $B' \in \beta$ such that $B \cup (-B) \subseteq B'$.

Interchanging if necessary h_n with $-h_n$ ($\in B'$), we will have

$$\langle x_n^* - x_0^*, h_n \rangle > 2\varepsilon. \tag{6}$$

From the β-differentiability of f at x, there exists $\delta > 0$, such that $B[x, \delta m] \subseteq D$, where $m > 0$ is chosen such that $B' \subseteq B[0, m]$, and

$$f(x + th) - f(x) - \langle x^*, th \rangle \leq t\varepsilon, \quad (t \in (0, \delta],\ h \in B').$$

Hence

$$f(x + th_n) - f(x) - \langle x^*, th_n \rangle \leq t\varepsilon, \quad (n \in \mathbb{N},\ t \in (0, \delta]). \tag{7}$$

Using the fact that $x_n^* \in \partial f(x_n)$, one obtains $\langle x_n^*, x + \delta h_n - x_n \rangle \leq f(x + \delta h_n) - f(x_n)$, hence

$$\langle x_n^*, \delta h_n \rangle \leq f(x + \delta h_n) - f(x) + \langle x_n^*, x_n - x_n \rangle + f(x) - f(x_n). \tag{8}$$

From (6), (7) and (8) we have

$$\begin{aligned} 2\varepsilon\delta &< \langle x_n^* - x_0^*, \delta h_n \rangle \\ &= \langle x_n^*, \delta h_n \rangle - \langle x_0^*, \delta h_n \rangle \\ &\leq f(x + \delta h_n) - f(x) + \langle x_n^*, x_n - x_n \rangle + f(x) - f(x_n) - \langle x_0^*, \delta h_n \rangle \\ &= (f(x + \delta h_n) - f(x) - \langle x_0^*, \delta h_n \rangle) + \langle x_n^*, x_n - x_n \rangle + f(x) - f(x_n) \\ &\leq \varepsilon\delta + \|x_n^*\| \|x_n - x_n\| + |f(x) - f(x_n)|. \end{aligned}$$

The convex function f being continuous, it is locally Lipschitz, hence the sequence $\|x_n^*\|$ is bounded (by the Lipschitz constant). For $n \to \infty$ one obtains $2\varepsilon\delta \leq \varepsilon\delta$, a contradiction. □

THEOREM 7. *Let E be a normed space, β a bornology, f a continuous convex function on an open convex set $D \subseteq E$. Then the following statements are equivalent:*

(i) *f is β-differentiable at $x \in D$.*

(ii) *Each selection $\varphi : D \to E^*$ for the subdifferential ∂f is $\tau_{\|\cdot\|}$-τ_β continuous at $x \in D$.*

(iii) *There exists a selection $\varphi : D \to E^*$ for the subdifferential ∂f which is $\tau_{\|\cdot\|}$-τ_β continuous at $x \in D$.*

Proof. (i)⇒(ii). According to the previous proposition, ∂f is $\tau_{\|\cdot\|}$-τ_β u.s.c., hence each of its selections will be $\tau_{\|\cdot\|}$-τ_β continuous.

(ii)⇒(iii). This implication is obvious.

(iii)⇒(i). For $y \in D$ we have $\langle \varphi(x), y - x \rangle \leq f(y) - f(x)$, because $\varphi(x) \in \partial f(x)$. Using $\varphi(y) \in \partial f(x)$ one obtains $\langle \varphi(y), x - y \rangle \leq f(x) - f(y)$, hence:

$$0 \leq f(y) - f(x) - \langle \varphi(x), y - x \rangle \leq \langle \varphi(y) - \varphi(x), y - x \rangle. \tag{9}$$

For $h \in E, t > 0$, replacing in (9) $y = x + th$, dividing by t and letting $t \to 0+$, one obtains $0 \leq \mathrm{d}^+ f(x)(h) - \varphi(x)(h) \leq 0$, hence

$$\mathrm{d}^+ f(x) = \varphi(x) \in E^*,$$

and f is G-differentiable at x.

For $B \in \beta$, $h \in B$, $t > 0$, $y = x + th$, we have

$$0 \leq \frac{1}{t}(f(x+th) - f(x)) - \mathrm{d}f(x) \leq \langle \varphi(x+th) - \varphi(x), h \rangle .$$

From the $\tau_{\|\cdot\|}$-τ_β continuity of φ, the right hand side tends to 0 uniformly for $h \in B$ (=bounded) as $t \to 0+$, and the conclusion follows. □

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