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CONDENSATION OF THE SINGULARITIES IN THE THEORY OF OPERATOR IDEALS

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. In the present paper there are given some applications of the principle of condensation of the singularities of families of nonnegative functions established by W. W. Breckner in 1984. They reveal Baire category information on certain subsets of a normed linear space X of the second category that are defined by means of an inequality of the type $f(x) < \infty$, where f is a given function from X to $[0, \infty]$. Sets of this type occur frequently in the theory of operator ideals. They are constructed individually by using entropy or approximation numbers of operators. By specializing the general results given in the paper it follows that such operator sets are of the first category, while their complements are residual G_δ-sets, of the second category, uncountable and dense.

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1. INTRODUCTION

In functional analysis concrete normed or quasinormed linear spaces are introduced frequently as sets consisting of all the elements x of a certain normed linear space X that satisfy an inequality $f(x) < \infty$, where f is a given function from X to $[0, \infty]$. For instance, the classical sequence spaces l_p are of this type, and so are numerous linear spaces occuring in the theory of operator ideals: $l_{p,q}$, $l_{p,\infty}$, $L^s_{p,q}(E_0, E)$, $L^e_{p,q}(E_0, E)$ etc. (see [3], [6]). The aim of our present paper is to look at such linear spaces from a topological point of view and to prove that such linear spaces are often subsets of the first category of the space X comprising them, while their complements are dense in X. Investigations of this kind have already been carried out in [2] in the cases when X is the normed linear space $\mathbf{L}(E)$ of all continuous linear operators on an infinite-dimensional Banach space E or the normed linear space $\mathbf{b}$ consisting of all bounded sequences of real or complex numbers. Here we continue these

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investigations in a more general framework, but just like in [2] we use the following *principle of condensation of the singularities of families of nonnegative functions* as main tool.

THEOREM 1. *Let X be a topological linear space, and let F be a family of lower semicontinuous functions from X to $[0,\infty]$ satisfying the following conditions:*

(i) *there exists a number $r \in [1,\infty]$ such that for all $f \in F$ one has*

$$f(x-y) \leq r \max\{f(x), f(y)\}$$

whenever $x, y \in X$ and $f(x) < \infty$, $f(y) < \infty$;

(ii) *there exists a bounded subset M of X such that*

$$\sup\{f(x)|\ (f,x) \in F \times M\} = \infty.$$

Then the set S_F of all singularities of F, i.e., the set consisting of all $x \in X$ for which $\sup\{f(x)|\ f \in F\} = \infty$, *has the following properties:*

1°. *S_F is a residual G_δ-set.*

2°. *If in addition X is of the second category, then S_F is of the second category and dense in X.*

3°. *If in addition X is of the second category, satisfies the separation axiom T_1 and has nonzero elements, then S_F is of the second category, uncountable and dense in X.*

This theorem is a generalization of the classical principle of condensation of the singularities of a family of continuous linear operators from a Banach space into another normed linear space. It follows from the results stated in [1].

Throughout the paper $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, while $\mathbb{N}$ is the set of all positive integers. All linear spaces that will be used are over $\mathbb{K}$. The set consisting of all nonnegative real numbers is denoted by $\mathbb{R}_+$.

2. A FIRST GROUP OF APPLICATIONS OF THE PRINCIPLE OF CONDENSATION OF THE SINGULARITIES

THEOREM 2. *Let p and q be positive real numbers, let X be a normed linear space of the second category, and let s be a map that assigns to each $x \in X$ a sequence $s(x) := (s_n(x))$ of real numbers such that the following conditions are satisfied:*

(i) $\|x\| = s_1(x) \geq \cdots \geq s_n(x) \geq \cdots \geq 0$, *for all $x \in X$;*

(ii) $s_{m+n-1}(x-y) \leq s_n(x) + s_n(y)$, *for all $m, n \in \mathbb{N}$ and all $x, y \in X$;*

(iii) *there exists an $x_0 \in X$ such that* $\inf\{s_n(x_0)|\ n \in \mathbb{N}\} > 0$.

Then the set $X^s_{p,q}$ of all $x \in X$ for which

$$\sum_{n=1}^{\infty} n^{q/p-1}[s_n(x)]^q < \infty$$

has the following properties:

1°. *$X \setminus X^s_{p,q}$ is a residual G_δ-set, of the second category, uncountable and dense in X.*

2°. *$X^s_{p,q}$ is a subset of the first category in X.*

Proof. 1°. To simplify the notation we put $r := q/p - 1$. Next we define for each $m \in \mathbb{N}$ the function $f_m : X \to \mathbb{R}_+$ by

$$f_m(x) := \sum_{n=1}^{m} n^r [s_n(x)]^q.$$

We claim that all the functions f_m ($m \in \mathbb{N}$) are continuous. Indeed, from (i) and (ii) it follows that

$$s_n(x+y) \le s_n(x) + \|x\|, \quad \text{for all } n \in \mathbb{N} \text{ and all } x, y \in X.$$

This property implies

$$|s_n(x) - s_n(y)| \le \|x - y\|, \quad \text{for all } n \in \mathbb{N} \text{ and all } x, y \in X.$$

Consequently, for every $n \in \mathbb{N}$ the function

$$\forall\, x \in X \mapsto s_n(x) \in \mathbb{R} \tag{1}$$

is continuous. This result implies that all the functions f_m ($m \in \mathbb{N}$) are continuous.

Next we put $a := \max\{1, 2^r\}$ and prove that the following inequality is valid for all $m \in \mathbb{N}$:

$$f_m(x-y) \le a 2^{q+2} \max\{f_m(x), f_m(y)\}, \quad \text{whenever } x, y \in X. \tag{2}$$

To this end we fix $m \in \mathbb{N}$ as well as $x, y \in X$. Further we choose a $k \in \mathbb{N}$ such that $2(k-1) < m \le 2k$. It is easily seen that for all $n \in \mathbb{N}$ the following inequalities hold:

$$\max\{(2n-1)^r, (2n)^r\} \le a n^r;$$

$$[s_{2n-1}(x-y)]^q \le [s_n(x) + s_n(y)]^q \le 2^q([s_n(x)]^q + [s_n(y)]^q).$$

They imply

$$\begin{aligned} f_m(x-y) &\le f_{2k}(x-y) \le \sum_{n=1}^{k} [(2n-1)^r + (2n)^r][s_{2n-1}(x-y)]^q \\ &\le a 2^{q+1} \Big\{ \sum_{n=1}^{k} n^r [s_n(x)]^q + \sum_{n=1}^{k} n^r [s_n(y)]^q \Big\} \\ &= a 2^{q+1} \{f_k(x) + f_k(y)\} \le a 2^{q+2} \max\{f_k(x), f_k(y)\}. \end{aligned}$$

Taking into account that $k \le m$ we get

$$f_m(x-y) \le a^{q+2} \max\{f_m(x), f_m(y)\}.$$

Consequently (2) holds for all $m \in \mathbb{N}$.

Next we select an $x_0 \in X$ for which

$$b := \inf \{s_n(x_0)|\ n \in \mathbb{N}\} > 0. \tag{3}$$

Then we have

$$f_m(x_0) \geq (1^r + \cdots + m^r)b, \quad \text{for all } m \in \mathbb{N}.$$

Since the series $\sum n^r$ diverges, it follows that

$$\sup \{f_m(x_0)|\ m \in \mathbb{N}\} = \infty. \tag{4}$$

Summing up our considerations we can conclude that the family $F := \{f_m|\ m \in \mathbb{N}\}$ consists of continuous functions from X to $\mathbb{R}_+$ that satisfy (2) and (4). In view of the assertions 1° and 3° in Theorem 1 the set S_F of all singularities of F is a residual G_δ-set, of the second category, uncountable and dense in X. But, on the other hand, we have $S_F = X \setminus X^s_{p,q}$. Consequently the assertion 1° of our theorem is true.

2°. This assertion follows immediately from assertion 1°. □

Theorem 3. *Let p be a positive real number, let X be a normed linear space of the second category, and let s be a map that assigns to each $x \in X$ a sequence $s(x) := (s_n(x))$ of real numbers satisfying the conditions* (i)–(iii) *in* Theorem 2. *Then the set $X^s_{p,\infty}$ of all $x \in X$ for which*

$$\sup \{n^{1/p} s_n(x)|\ n \in \mathbb{N}\} < \infty$$

has the following properties:

1°. *$X \setminus X^s_{p,\infty}$ is a residual G_δ-set, of the second category, uncountable and dense in X.*

2°. *$X^s_{p,\infty}$ is a subset of the first category in X.*

Proof. 1°. For each $m \in \mathbb{N}$ we define the function $f_m : X \to \mathbb{R}_+$ by

$$f_m(x) := \max \{1^{1/p} s_1(x), \ldots, m^{1/p} s_m(x)\}.$$

Since for every $n \in \mathbb{N}$ the function (1) is continuous, it follows that all the functions f_m ($m \in \mathbb{N}$) are continuous.

Next we put $a := 2^{1+1/p}$ and prove that the following inequality holds for all $m \in \mathbb{N}$:

$$f_m(x - y) \leq a \max \{f_m(x), f_m(y)\}, \quad \text{whenever } x, y \in X. \tag{5}$$

To this end we fix $m \in \mathbb{N}$ as well as $x, y \in X$. First we note that for all $n \in \mathbb{N}$ we have

$$(2n-1)^{1/p} s_{2n-1}(x - y) \leq (2n)^{1/p} [s_n(x) + s_n(y)] \leq 2^{1/p} \{f_m(x) + f_m(y)\}$$

when $2n - 1 \leq m$, and

$$\begin{aligned} (2n)^{1/p} s_{2n}(x - y) &\leq (2n)^{1/p} s_{2n-1}(x - y) \\ &\leq (2n)^{1/p} [s_n(x) + s_n(y)] \\ &\leq 2^{1/p} \{f_m(x) + f_m(y)\} \end{aligned}$$

when $2n \le m$. This result implies

$$f_m(x-y) \le 2^{1/p}\{f_m(x)+f_m(y)\}.$$

Consequently, (5) is valid.

Next we choose an $x_0 \in X$ for which (3) holds. Then we have $f_m(x_0) \ge m^{1/p}b$ for all $m \in \mathbb{N}$, whence (4) follows.

Summing up our considerations we can conclude that the family $F := \{f_m \mid m \in \mathbb{N}\}$ consists of continuous functions from X to $\mathbb{R}_+$ that satisfy (5) and (4). By applying the assertions 1° and 3° of Theorem 1 we conclude that the set S_F of all singularities of F is a residual G_δ-set, of the second category, uncountable and dense in X. Taking into account that $S_F = X \setminus X^s_{p,\infty}$, it follows that the assertion 1° of our theorem is true.

2°. This assertion follows immediately from assertion 1°. □

Theorems 2 and 3 provide a series of results revealing Baire category information on sets occurring in the theory of operator ideals. To illustrate this let E_0 and E be normed linear spaces. By $L(E_0, E)$ we denote the linear space of all continuous linear operators $T : E_0 \to E$ endowed with the norm

$$\|T\| := \sup\{\|T(x)\| \mid x \in E_0,\ \|x\| \le 1\}.$$

Further we denote by $K(E_0, E)$ the set consisting of all compact linear operators $T : E_0 \to E$. It is well-known that $K(E_0, E) \subseteq L(E_0, E)$.

Let B_0 and B be the closed unit balls in E_0 and E, respectively. Given any $T \in L(E_0, E)$, we put

$$e_n(T) := \inf \Big\{\alpha \in \mathbb{R}_+ \mid \exists\, m \in \mathbb{N}, \exists\, y_1, \ldots, y_m \in E :\ m \le 2^{n-1},$$
$$T(B_0) \subseteq (y_1 + \alpha B) \cup \cdots \cup (y_m + \alpha B)\Big\}$$

for all $n \in \mathbb{N}$. The number $e_n(T)$ is called the *nth outer entropy number of* T (see [5, p. 168]).

Theorem 4. *Let p and q be positive real numbers, let E_0 and E be normed linear spaces such that $L(E_0, E) \setminus K(E_0, E) \ne \emptyset$ and E is complete. Then the set $L^e_{p,q}(E_0, E)$ of all $T \in L(E_0, E)$ for which*

$$\sum_{n=1}^{\infty} n^{q/p-1}[e_n(T)]^q < \infty$$

has the following properties:

1°. *$L(E_0, E) \setminus L^e_{p,q}(E_0, E)$ is a residual G_δ-set, of the second category, uncountable and dense in $L(E_0, E)$.*
2°. *$L^e_{p,q}(E_0, E)$ is a subset of the first category in $L(E_0, E)$.*

Proof. Since E is complete, $L(E_0, E)$ is also complete, and hence of the second category.

Recall that the outer entropy numbers have the following properties (see [5, pp. 168–169]):

$$\|T\| = e_1(T) \geq \cdots \geq e_n(T) \geq \cdots \geq 0, \quad \text{for all } T \in L(E_0, E);$$

$$e_{m+n-1}(T-U) \leq e_m(T) + e_n(U), \quad \text{for all } m, n \in \mathbb{N} \text{ and all } T, U \in L(E_0, E).$$

Moreover, an operator $T \in L(E_0, E)$ is compact if and only if

$$\lim_{n\to\infty} e_n(T) = 0.$$

In view of these properties Theorem 2 is applicable. □

By analogy with the deduction of Theorem 4 from Theorem 2, Theorem 3 can also be specialized for the case when $X := L(E_0, E)$ and $s(x)$ is the sequence of outer entropy numbers.

By taking $X := L(E_0, E)$, but choosing a sequence of additive s-numbers instead of the sequence of outer entropy numbers (for instance, the sequence of approximation numbers, the sequence of Kolmogorov numbers, or the sequence of Gelfand numbers), we can obtain quite similarly further specializations of the Theorems 2 and 3 to classes of linear operators occuring in the theory of operator ideals (see [5], [6]). We leave the details to the reader. Obviously, Theorem 2.1 given in [2] can be obtained in this manner.

3. A SECOND GROUP OF APPLICATIONS OF THE PRINCIPLE OF CONDENSATION OF THE SINGULARITIES

THEOREM 5. *Let X be a normed linear space of the second category, let s be a map that assigns to each $x \in X$ a sequence $s(x) := (s_n(x))$ of real numbers satisfying the conditions* (i)–(iii) *in Theorem* 2*, and let $\phi := (\phi_n)$ be a sequence of functions $\phi_n : \mathbb{R}^n \to \mathbb{R}$ such that the following properties are satisfied:*

(j) *ϕ_n is sublinear, i.e., the inequality*

$$\phi_n(\alpha t_1 + \beta u_1, \ldots, \alpha t_n + \beta u_n) \leq \alpha\phi_n(t_1, \ldots, t_n) + \beta\phi_n(u_1, \ldots, u_n)$$

holds whenever $\alpha, \beta \in \mathbb{R}_+$ and $(t_1, \ldots, t_n), (u_1, \ldots, u_n) \in \mathbb{R}^n$;

(jj) *$\phi_n(t_1, \ldots, t_n) \geq 0$, for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$;*

(jjj) *$\phi_n(t_1, \ldots, t_n) = \phi(|t_{\sigma(1)}|, \ldots, |t_{\sigma(n)}|)$, for any $(t_1, \ldots, t_n) \in \mathbb{R}^n$ and any permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$;*

(jv) *$\phi_n(1, 0, \ldots, 0) = 1$;*

(v) *$\sup\{\phi_n(1, \ldots, 1) \mid n \in \mathbb{N}\} = \infty$.*

Then the set X^s_ϕ of all $x \in X$ for which

$$\sup\{\phi_n(s_1(x), \ldots, s_n(x)) \mid n \in \mathbb{N}\} < \infty$$

has the following properties:

1°. *$X \setminus X^s_\phi$ is a residual G_δ-set, of the second category, uncountable and dense in X.*

2°. *X^s_ϕ is a subset of the first category in X.*

Proof. 1°. For each $n \in \mathbb{N}$ we define the function $f_n : X \to \mathbb{R}_+$ by

$$f_n(x) = \phi_n(s_1(x), \dots, s_n(x)).$$

In view of the continuity of the functions ϕ_n (because they are sublinear) and (1) (see the proof of Theorem 2), it follows that all the functions f_n ($n \in \mathbb{N}$) are continuous.

Next we prove that the following inequality is valid for all $n \in \mathbb{N}$:

$$f_n(x-y) \leq 4 \max\{f_n(x), f_n(y)\} \text{ whenever } x, y \in X. \tag{6}$$

To this end we fix $n \in \mathbb{N}$ as well as $x, y \in X$. Then we have

$$\begin{aligned}\sum_{i=1}^{k} s_i(x-y) &\leq \sum_{i=1}^{2k} s_i(x-y)\\ &= \sum_{i=1}^{k} s_{2i-1}(x-y) + \sum_{i=1}^{k} s_{2i}(x-y)\\ &\leq 2\sum_{i=1}^{k} s_{2i-1}(x-y)\\ &\leq 2\sum_{i=1}^{k}[s_i(x)+s_i(y)]\end{aligned}$$

for every $k \in \{1, \dots, n\}$. By applying a lemma due to Ky Fan (see [4, p. 97, Lemma 3.1]), it follows that

$$\phi_n(s_1(x-y), \dots, s_n(x-y)) \leq \phi_n(2[s_1(x)+s_1(y)], \dots, 2[s_n(x)+s_n(y)]).$$

Consequently we have

$$f_n(x-y) \leq 2f_n(x) + 2f_n(y) \leq 4\max\{f_n(x), f_n(y)\}.$$

Now we select an $x_0 \in X$ for which (3) holds. Then we have

$$f_n(x_0) \geq \phi_n(b, \dots, b) = b\phi_n(1, \dots, 1), \quad \text{for all } n \in \mathbb{N},$$

whence

$$\sup\{f_n(x_0) |\ n \in \mathbb{N}\} = \infty. \tag{7}$$

Summing up our considerations we can conclude that the family $F := \{f_n |\ n \in \mathbb{N}\}$ consists of continuous functions from X to $\mathbb{R}_+$ that satisfy (6) and (7). In view of the assertions 1° and 3° in Theorem 1 the set S_F of all singularities of F is a residual G_δ-set, of the second category, uncountable and dense in X. But, on the other hand, we have $S_F = X \setminus X^s_\phi$. Consequently the assertion 1° of our theorem is true.

2°. This assertion follows immediately from assertion 1°. □

By specializing X, s and ϕ it is easy to construct different sets of the first category by means of Theorem 5. For instance, certain sets occuring in the theory of operator ideals are obtained when X is replaced by the normed linear space $L(E_0, E)$ of all continuous linear operators from a normed linear space E_0 into another normed linear space E, s is defined by using the outer entropy numbers or additive s-numbers, and ϕ is a sequence of symmetric gauge functions (see [7, pp. 84–92]) or a symmetric norming function (see [4, pp. 95–105], [8]–[11]).

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