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BERSTEIN-SCHURER BIVARIATE OPERATORS

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Abstract. The sequence of bivariate operators of Bernstein-Schurer is constructed and some approximation properties of this sequence are studied.

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1. PRELIMINARIES

In 1962, Schurer F., (see [5]) introduced and studied the linear positive operators $\widetilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$, defined for any function $f \in C([0,1+p])$ by:

$$\left(\widetilde{B}_{m,p}\, f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x)\, f\, \left(\tfrac{k}{m}\right) \tag{1}$$

where

$$\widetilde{p}_{m,k}(x) = \tbinom{m+p}{k} x^k (1-x)^{m+p-k} \tag{2}$$

are the so called "fundamental polynomials" of Bernstein-Schurer. Note that in (1) and (2), p denotes a non-negative fixed integer. Clearly, for $p = 0$ the operators $\widetilde{B}_{m,0}$ are the classical operators of Bernstein. Some special properties of the polynomials (2) are established in our paper [2]. In the paper [3] we discussed about the remainder operator associated to $\widetilde{B}_{m,p}$.

The most important approximation properties of the operators $\widetilde{B}_{m,p}$ are contained in the following theorem

THEOREM 1. (see [1], [5]). *The Bernstein-Schurer operators defined at (1) have the following properties:*

(i)

$$\begin{aligned}
&(\widetilde{B}_{m,p}\, e_0)(x) = 1,\\
&(\widetilde{B}_{m,p}\, e_1)(x) = (1 + p/m)x,\\
&(\widetilde{B}_{m,p}\, e_2)(x) = (m+p)\, m^{-2}\{(m+p)x^2 + x(1-x)\},
\end{aligned}$$

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for any $x \in [0, 1+p]$, *where* $e_i(x) = x^i$, $(i = 0, 1, 2)$ *are the test functions;*

(ii) $\lim_{m\to\infty} \widetilde{B}_{m,p}\, f = f$, *uniformly on* $[0, 1+p]$, $(\forall)\ f \in C([0, 1+p])$;

(iii) $\left|(\widetilde{B}_{m,p}\, f)(x) - f(x)\right| \le 2\omega_f\left(\delta_{m,p,x}\right)$,
where ω_f *denotes the first order modulus of smoothness and*

$$\delta_{m,p,x} = \sqrt{p^2x^2 + (m+p)\, x\frac{1-x}{m}}.$$

The purposes of the present paper are the following:

1^0. To construct the sequence of bivariate operators of Bernstein-Schurer;

2^0. To study the uniform convergence of the sequence of the approximants defined by this sequence to the approximated bivariate function f;

3^0. To estimate the order of the above approximation using the first order modulus of smoothness for bivariate functions.

We need the auxiliary results, contained in the following theorems

THEOREM 2. (see [7]). *Let* $I, J \subseteq \mathbb{R}$ *be two compact intervals of the real axis and let* $(L_{m,n})_{m,n\in\mathbb{N}}$ *be a sequence of positive linear operators defined on* $C(I \times J)$.

If $e_{ij}(x, y) = x^iy^j$ $(0 \le i + j \le 2\,,\ \ i, j \in \mathbb{N})$ *are the test functions defined on* $I \times J$ *and*

$$\lim_{m,n\to\infty} (L_{m,n}\, e_{ij})(x, y) = e_{ij}(x, y) \tag{3}$$

uniformly on $I \times J$, *then*

$$\lim_{m,n\to\infty} (L_{m,n}\, f)(x, y) = f(x, y) \tag{4}$$

uniformly on $I \times J$, *for any* $f \in C(I \times J)$.

Note that the Theorem 2 is the well known Korovkin-type theorem for the approximation of bivariate functions.

THEOREM 3. (see [7]). *Let* $I, J \subseteq \mathbb{R}$ *be two compact intervals of the real axis and let be* $f \in C(I, J)$. *The first order modulus of smoothness of the bivariate function* f *is the function* $\omega_f : \mathbb{R}_+^2 \to \mathbb{R}_+$, *defined by*

$$\omega_f(\delta_1, \delta_2) = \max\{|f(x', y') - f(x, y)| : |x - x'| \le \delta_1, |y - y'| \le \delta_2\}. \tag{5}$$

Let $(L_{m,n})_{m,n\in\mathbb{N}}$ *be a sequence of linear positive operators defined on* $C(I \times J)$. *If* $L_{m,n}(1; x, y) = 1$, *the following inequality:*

$$\begin{aligned} &|(L_{m,n}\, f(x, y) - f(x, y)| \le \\ &\le \left\{1 + (\delta_1)^{-1}\left\{L_{m,n}((\circ - x)^2; x, y)\right\}^{\frac{1}{2}} + (\delta_2)^{-1}\left\{L_{m,n}((* - y)^2; x, y)\right\}^{\frac{1}{2}}\right. \\ &\left. +(\delta_1\delta_2)^{-1}\left\{L_{m,n}((\circ - x)(* - y); x, y)\right\}^{1/2}\right\}\omega_f(\delta_1, \delta_2) \end{aligned} \tag{6}$$

holds for any $f \in C(I \times J)$, *any* $(x, y) \in I \times J$ *and any* $\delta_1 > 0$, $\delta_2 > 0$.

2. MAIN RESULTS

Let p, q be two non-negative given integers and let $\widetilde{B}_{m,p} : C([0, 1 + p]) \to C([0, 1])$, $\widetilde{B}_{n,q} : C([0, 1 + q]) \to C([0, 1])$ be the Bernstein-Schurer operators defined at (1). Let f be a bivariate function $f \in C([0, 1 + p] \times [0, 1 + q])$. Then the "parametric extensions" (for this terminus see Delvos I., and Schempp W., [4]) of the operators $\widetilde{B}_{m,p}$, $\widetilde{B}_{n,q}$ are defined as follows

$$\left(\widetilde{B}^x_{m,p}\ f\right)(x, y) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\tfrac{k}{m}, y\right) \quad (y \in [0, 1 + q]), \tag{7}$$

$$\left(\widetilde{B}^x_{n,q}\ f\right)(x, y) = \sum_{j=0}^{n} \widetilde{p}_{n,j}(y) f(x, \tfrac{j}{n}), \quad (x \in [0, 1 + p]). \tag{8}$$

By direct computation one obtain the results contained in the following two theorems.

THEOREM 4. *The parametric extensions of the Bernstein-Schurer commute on* $C([0, 1 + p] \times [0, 1 + q])$. *Their product is the linear positive operator*

$$\widetilde{B}_{m,n,p,q} : C([0, 1 + p] \times [0, 1 + q]) \to C([0, 1] \times [0, 1]),$$

which associates to any function $f \in C([0, 1 + p] \times [0, 1 + q])$ *the approximant*

$$\left(\widetilde{B}_{m,n,p,q}\ f\right)(x, y) = \sum_{k=0}^{m+p} \sum_{j=o}^{n+q} \widetilde{p}_{m,n}(x) \widetilde{p}_{n,j}(y)\ f\left(\tfrac{k}{m}, \tfrac{j}{n}\right), \tag{9}$$

where $\widetilde{p}_{m,n}(x)$, $\widetilde{p}_{n,j}(y)$ *are the fundamental polynomials of Bernstein-Schurer.*

THEOREM 5. *The bivariate Bernstein-Schurer operator (9) interpolates the function* $f \in C([0, 1 + p] \times [0, 1 + q])$ *in the point* $(0, 0)$, *i.e.*

$$\left(\widetilde{B}_{m,n,p,q}\ f\right)(0, 0) = f(0, 0). \tag{10}$$

THEOREM 6. *If* $e_{ij}(x, y) = x^i y^j$ $(0 \le i + j \le 2,\ i, j \in \mathbb{N})$ *are the test functions, the following equalities:*

(i) $(\widetilde{B}_{m,n,p,q}\ e_{00})(x, y) = 1$;
(ii) $(\widetilde{B}_{m,n,p,q}\ e_{10})(x, y) = \{1 + p/m\}x$;
(iii) $(\widetilde{B}_{m,n,p,q}\ e_{01})(x, y) = \{1 + q/n\}y$;
(iv) $(\widetilde{B}_{m,n,p,q}\ e_{11})(x, y) = \{1 + p/m\} \times \{1 + q/n\}xy$;
(v) $(\widetilde{B}_{m,n,p,q}\ e_{20})(x, y) = (m + p)\, m^{-2}\{(m + p)x^2 + x(1 - x)\}$;
(vi) $(\widetilde{B}_{m,n,p,q}\ e_{02})(x, y) = (n + q)\, n^{-2}\{(n + q)y^2 + y(1 - y)\}$;

hold, for any $(x, y) \in [0, 1 + p] \times [0, 1 + q]$.

Proof. We shall prove only the identity (iii), the proofs of the others being similar.

$$\begin{aligned}\left(\widetilde{B}_{m,n,p,q}\ e_{11}\right)(x,y) &= \sum_{k=0}^{m+p}\sum_{j=0}^{n+q}\widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)\cdot(k/m)(j/n)\\ &=\left\{\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)(k/m)\right\}\left\{\sum_{j=0}^{n+q}\widetilde{p}_{n,j}(y)(j/n)\right\}\\ &=\left(\widetilde{B}_{m,p}\ e_1(x)\right)\left(\widetilde{B}_{n,q}\ e_1(y)\right)\\ &=(1+p/m)(1+q/n)xy. \qquad \square\end{aligned}$$

THEOREM 7. *The sequence* $\left\{\widetilde{B}_{m,n,p,q}\ f\right\}_{m,n\in\mathbb{N}}$ *converges to* f*, uniformly on* $C([0,1+p]\times[0,1+q])$*, for any* $f\in C([0,1+p]\times[0,1+q])$*.*

Proof. Using Theorem 6, it follows that $\widetilde{B}_{m,n,p,q}\ e_{ij}$ converges to e_{ij} uniformly on $C([0,1+p]\times[0,1+q])$, for any $i,j\in\mathbb{N}$ satisfying the conditions $0\le i+j\le 2$. Next, applying the Theorem 2 we get the desired result. □

THEOREM 8. *The following inequality*

$$\left|(\widetilde{B}_{m,n,p,q}\ f)(x,y)-f(x,y)\right|\le 4\omega_f(\delta_{m,p,x},\delta_{n,q,y}) \tag{11}$$

holds, for any $f\in C([0,1+p]\times[0,1+q])$*, and any* $(x,y)\in[0,1+p]\times[0,1+q]$*, where:*

$$\delta_{m,p,x}=\sqrt{p^2m^{-2}x^2+(m+p)m^{-2}x(1-x)}\,, \tag{12}$$

$$\delta_{n,q,y}=\sqrt{q^2n^{-2}y^2+(n+q)n^{-2}y(1-y)}\,. \tag{13}$$

Proof. It is easy to observe that

$$\begin{aligned}\widetilde{B}_{m,n,p,q}((\circ-x)^2;x,y)&=p^2m^{-2}x^2+(m+p)m^{-2}x(1-x),\\ \widetilde{B}_{m,n,p,q}((*-y)^2;x,y)&=q^2n^{-2}y^2+(n+q)n^{-2}y(1-y).\end{aligned}$$

Next, using the Theorem 3 we arrive to the desired inequality (11). □

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