THE ABSTRACT, MULTIDIMENSIONAL VARIETIES 
AND THEIR CLASSIFICATION

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday

Abstract. We define abstract multidimensional variety without borders, using the investigation of the complex of multi-ary relations (H. Martini and P. Soltan, 2003 [3]) and the notion of compact, combinatorial, multidimensional variety without borders (V. G. Boltyanski and V. A. Efrimovici, 1982 [1]). We indicate the classification of this kind of varieties similarly to the results of classification of compact, two-dimensional surfaces without borders (V. G. Boltyanski and V. A. Efrimovici, 1982 [1]). We use varieties’ genders (modulo Euler characteristic (V. G. Boltyanski, 1995 [2])) to classify them.

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Let $K^n = \{S^0, S^1, \ldots, S^n\}$ be a complex of multi-ary relations [3].

**Definition 1.** If the complex $K^n$ satisfies the conditions:
(1) $\forall S^{n-1} \subset S^{n-1}$ is a commune face with exactly two simplexes from $S^n$;
(2) For $\forall S^n_i, S^n_j \subset S^n, i \neq j$, there is a sequence
   $$S^n_i = S^n_1, S^n_2, \ldots, S^n_q = S^n_j$$
   from $S^n$ so that the pair $S^n_{ik}, S^n_{ik+1}, 1 \leq k \leq q-1$ satisfies the relation
   $S^n_{ik} \cap S^n_{ik+1} \in S^{n-1}$;
(3) $S^m \in K^n, 0 \leq m \leq n$, is at least a face of one simplex $S^n \in K^n$;
(4) For $\forall S^n_i, S^n_j \subset S^n, i \neq j$, so that $S^n_i \cap S^n_j = S^m \in S^m$, the sequence
   from 2. involve the relation $S^m \in S^n_{i1} \cap S^n_{i2} \cap \ldots \cap S^n_{i_l}$,
then $K^n$ is called abstract variety of dimension $n$ and without borders, that is denoted by $V^n$.

Let $Z$ be the group of integer numbers, $f : V^n \rightarrow Z$ – a single-valued map that satisfies: for $\forall S^m \in S^n, f(-S^m) = -f(S^m)$, where $0 \leq m \leq n$. We consider the group of chains of dimension $m$ of the complex $K^n$ and $\forall m \in ...
\[ L^m \implies l^m = g_1 S_1^m + g_2 S_2^m + \ldots + g_{\alpha_m} S_{\alpha_m}^m, \]
where \( g_i \in \mathbb{Z}, i = 1, \ldots, \alpha_m, \alpha_m = \text{card} S^m. \)

**Definition 2.** Let \( V^n \) be an abstract variety. If \( \exists l^m \in L^n, \Delta l = 0, \)
then \( V^n \) is said to be *oriented* variety, otherwise it is called *nonoriented* variety. The chain \( l^n \in L^n \) is said to be *cycle of dimension* \( m \) [3] of the complex \( K^n \) if \( \Delta l^n = 0. \) It is denoted by \( z^n. \)

**Theorem 3.** \( \forall z^n \in V^n \) has a unique representation by the formula: \( f(z^n) = g_1 S_1^m + g_2 S_2^m + \ldots + g_{\alpha_n} S_{\alpha_n}^m, \) where \( g_i = \pm 1, i \in \{1, \ldots, \alpha_n\}. \)

The spherical variety of dimension \( n \) will be denoted by \( V^n \) or \( \Sigma^n. \) It satisfies one of the relations: \( \chi(V^n) = 2, \) if \( n \) is even, or \( \chi(V^n) = 0, \) if \( n \) is odd.

**Theorem 4.** An abstract, oriented variety \( V^n \) is a spherical variety, if \( \forall V^{n-1} \subset V^n, \) where \( V^{n-1} \) is spherical, satisfies the relation:

\[ V^n \setminus V^{n-1} = K^n_1 \cup K^n_2, K^n_1 \cap K^n_2 = \emptyset \quad \text{and} \quad \chi(K^n_1) = \chi(K^n_2) = 1. \]

**Definition 5.** Let \( K^n \) be a complex of multary relations, \( S^k = [x_{i_0}, x_{i_1}, \ldots, x_{i_k}], k \in \{1, 2, \ldots, n\}, \) a simplex from \( K^n. \) We denote

\[ S^k = (x_{i_0}, x_{i_1}, \ldots, x_{i_k}) = S^k \setminus \{F_\lambda\}, \lambda \in \Lambda', \]
where \( \{F_\lambda: \lambda \in \Lambda'\} \) is the family of all faces of \( S^k. \) \( S^k \) is said to be *vacuum* of dimension \( k. \)

**Definition 6.** The variety \( V^n \) has \( t \) spherical borders of dimension \( n-1, \)
if \( S^m_1, S^m_2, \ldots, S^m_t \in V^n, S^m_i \cap S^m_j = \emptyset, i \neq j, i, j = 1, 2, \ldots, t. \)

Let \( \Sigma^n_1 \) and \( \Sigma^n_2 \) be two disjoint, isomorphic varieties that are generated by the sets \( X_1, X_2 \in F, X_1 \cap X_2 = \emptyset, \) where \( n \) is even. \( \text{Card} X_i = n + 1, i = 1, 2 \implies \exists \{S^m_1\}, \{S^m_2\}, \) that are respectively generated by the sets \( X_1 \) and \( X_2. \) So, we can take out from \( \Sigma^n_1 \) two vacuum of dimension \( n - 1. \) We denote them by \( S^m_{11} \) and \( S^m_{12} \) that are respectively suitable to the simplexes \( S^m_{11-1} \) and \( S^m_{12-1}. \) Isomorphically we take out \( S^n_{21} \) and \( S^n_{22} \) from \( \Sigma^n_2. \) We "stick" these borders to the isomorphic images from \( \Sigma^n_2. \) So we get the variety \( V^n. \) This is called the *variety of gender two*. We get inductively the countable set of varieties of respective gender:

\[(1) \]
\[ V^n_0, V^n_1, V^n_2, \ldots, V^n_p, \ldots \]

The construction of set (1) was done by the countable set of finite sets \( \{X_i\}_{i=1}^{\infty}, \) where \( X_i \in F, \forall i \geq 1. \) The set (1) satisfies the relation \( \chi(V^n_p) = 2 - 2p, \forall p \geq 0. \)

Let \( F' = F \setminus \bigcup_{i=1}^{\infty} X_i \) be a set of unused sets for (1). Similarly we construct one set of abstract, oriented varieties of odd dimension. In this case the
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varieties will be generated by the sets from $F'$. This set of varieties satisfies the relation $\chi(V^m_n) = 0$, $\forall q \geq 0$. So, now, the Euler characteristic cannot be used as a criterion of classification.

The set of all $\Delta$-cycle of dimension $m$ [3] of the variety $V^n$, $m = 0, 1, \ldots, n$, with respect to the addition of $\Delta$-chains form a commutative group $Z^m(\Delta)$. There are two kind of $\Delta$-cycles of $V^n$:

(a) $\Delta I^m = \Delta z^m = 0$;
(b) $\Delta \Delta I^m = \Delta(\Delta I^m) = 0$; $\Delta I^m \neq 0$.

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(b) $\Delta \Delta l^m = \Delta(\Delta l^m) = 0$; $\Delta l^m \neq 0$.

The set of cycles of dimensions $m$ with the property (a) forms a commu-
tative group $Z^m_0(\Delta)$. Let $r_0, r_1, \ldots, r_n$ be the ranks of the groups of $\Delta$-homologies of the variety $V^n$, $Z^m(\Delta)/Z^m_0(\Delta) = \Delta^m(V^n, Z)$, $m = 1, 2, \ldots, n$. It is known [3] that

\begin{equation}
\chi(V^n_j) = \sum_{i=0}^{n}(\pm 1)^i r_j, \ j \geq 0. \tag{2}
\end{equation}

So, we can classify the abstract, oriented varieties with odd dimension by comparing the sequences $(r_0^j, r_1^j, \ldots, r_n^j)$, $j \geq 0$.

**Definition 7.** If for the abstract, odd dimension varieties and without bor-
ders $V^n_1$ and $V^n_2$ the groups $\Delta^o(V^n_1, Z)$ and $\Delta^o(V^n_2, Z)$ are isomorphic, then $V^n_1$ and $V^n_2$ belong to the same class.

This classification establishes the set of oriented varieties of odd dimension:

\begin{equation}
V^n_0, V^n_1, V^n_2, \ldots, V^n_q, \ldots \tag{3}
\end{equation}

**Theorem 8.** Let $V^n$ be an arbitrary, abstract, oriented variety. There is one and only one element in (1) or (3), $V^n_p$ and $V^n_q$, so that $\chi(V^n) = \chi(V^n_p)$ or $\chi(V^n) = \chi(V^n_q)$.

Similarly it is constructed the set of abstract, nonoriented varieties:

\begin{equation}
V^n_1, V^n_2, \ldots, V^n_l, \ldots \tag{4}
\end{equation}

where $n \geq 2$ is even and $\chi(V^n_l) = 2 - l$.

**Theorem 9.** Let $V^n$ be an abstract, nonoriented variety, $n = 2m - 1 > 2$. There is one and only one element in (4), $V^n_l$, so that $\chi(V^n) = \chi(V^n_l)$.

So, the classification of abstract varieties of dimension $n$ is done by there genders (modulo Euler characteristic).

**REFERENCES**


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