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THE ABSTRACT, MULTIDIMENSIONAL VARIETIES AND THEIR CLASSIFICATION

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday

Abstract. We define abstract multidimensional variety without borders, using the investigation of the complex of multi-ary relations (H. Martini and P. Soltan, 2003 [3]) and the notion of compact, combinatorial, multidimensional variety without borders (V. G. Boltyanski and V. A. Efrimovici, 1982 [1]). We indicate the classification of this kind of varieties similarly to the results of classification of compact, two-dimensional surfaces without borders (V. G. Boltyanski and V. A. Efrimovici, 1982 [1]). We use varieties' genders (modulo Euler characteristic (V. G. Boltyanski, 1995 [2])) to classify them.

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Let $\mathcal{K}^n = \{\mathcal{S}^0, \mathcal{S}^1, \dots, \mathcal{S}^n\}$ be a complex of multi-ary relations [3].

DEFINITION 1. *If the complex $\mathcal{K}^n$ satisfies the conditions:*

(1) $\forall S^{n-1} \subset \mathcal{S}^{n-1}$ *is a commune face with exactly two simplexes from* $\mathcal{S}^n$*;*
(2) *For* $\forall S_i^n, S_j^n \in \mathcal{S}^n$, $i \neq j$, *there is a sequence*
$$S_{i_1}^n = S_1^n,\, S_{i_2}^n,\, \dots,\, S_{i_q}^n = S_j^n$$
from $\mathcal{S}^n$ *so that the pair* $S_{i_k}^n, S_{i_{k+1}}^n$, $1 \leq k \leq q-1$ *satisfies the relation* $S_{i_k}^n \cap S_{i_{k+1}}^n \in \mathcal{S}^{n-1}$*;*
(3) $S^m \in \mathcal{K}^n$, $0 \leq m \leq n$, *is at least a face of one simplex* $S^n \in \mathcal{K}^n$*;*
(4) *For* $\forall S_i^n, S_j^n \subset \mathcal{S}^n$, $i \neq j$, *so that* $S_i^n \cap S_j^n = S^m \in \mathcal{S}^m$, *the sequence from 2. involve the relation* $S^m \in S_{i_1}^n \cap S_{i_2}^n \cap \dots \cap S_{i_q}^n$,

then $\mathcal{K}^n$ is called ***abstract variety of dimension*** n ***and without borders****, that is denoted by* V^n.

Let $\mathbb{Z}$ be the group of integer numbers, $f : V^n \to \mathbb{Z}$ – a single-valued map that satisfies: for $\forall S^m \in \mathcal{S}^m$, $f(-S^m) = -f(S^m)$, where $0 \leq m \leq n$. We consider the group of chains of dimension m of the complex $\mathcal{K}^n$ and $\forall l^m \in$

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$\mathcal{L}^m \Longrightarrow$
$$l^m = g_1 S_1^m + g_2 S_2^m + \ldots + g_{\alpha_m}^m S_{\alpha_m}^m,$$
where $g_i \in \mathbb{Z}$, $i = 1, \ldots, \alpha_m$, $\alpha_m = \operatorname{card} \mathcal{S}^m$.

Definition 2. *Let V^n be an abstract variety. If $\exists l^n \in \mathcal{L}^n, \Delta l = 0$, then V^n is said to be* ***oriented*** *variety, otherwise it is called* ***nonoriented*** *variety. The chain $l^n \in \mathcal{L}^n$ is said to be* ***cycle of dimension m*** [3] *of the complex $\mathcal{K}^n$ if $\Delta l^n = 0$. It is denoted by z^n.*

Theorem 3. *$\forall z^n \in V^n$ has a unique representation by the formula: $f(z^n) = g_1 S_1^n + g_2 S_2^n + \ldots + g_{\alpha_n} S_{\alpha_n}^n$, where $g_i = \pm 1, i \in \{1, \ldots \alpha_n\}$.*

The spherical variety of dimension n will be denoted by V^n or Σ^n. It satisfies one of the relations: $\chi(V^n) = 2$, if n is even, or $\chi(V^n) = 0$, if n is odd.

Theorem 4. *An abstract, oriented variety V^n is a spherical variety, if $\forall V^{n-1} \subset V^n$, where V^{n-1} is spherical, satisfies the relation:*
$$V^n \backslash V^{n-1} = \mathcal{K}_1^n \cup \mathcal{K}_2^n,\ \mathcal{K}_1^n \cap \mathcal{K}_2^n = \emptyset \quad \textit{and}\ \chi(\mathcal{K}_1^n) = \chi(\mathcal{K}_2^n) = 1.$$

Definition 5. *Let $\mathcal{K}^n$ be a complex of multy-ary relations, $S^k = [x_{i_0}, x_{i_1}, \ldots, x_{i_k}], k \in \{1, 2, \ldots, n\}$, a simplex from $\mathcal{K}^n$. We denote*
$$\overset{\circ}{S}{}^k = (x_{i_0}, x_{i_1}, \ldots, x_{i_k}) = S^k \backslash \{F_\lambda\},\ \lambda \in \Lambda',$$
where $\{F_\lambda : \lambda \in \Lambda'\}$ is the family of all faces of S^k. $\overset{\circ}{S}{}^k$ is said to be ***vacuum*** *of dimension k.*

Definition 6. *The variety V^n has t* ***spherical borders*** *of dimension $n-1$, if $\exists \overset{\circ}{S}{}_1^n, \overset{\circ}{S}{}_2^n, \ldots, \overset{\circ}{S}{}_t^n \in V^n$, $S_i^n \cap S_j^n \neq \emptyset, i \neq j$, $i, j = 1, 2, \ldots, t$.*

Let Σ_1^n and Σ_2^n be two disjoint, isomorphic varieties that are generated by the sets $X_1, X_2 \in F$, $X_1 \cap X_2 = \emptyset$, where n is even. $\operatorname{Card} X_i > n + 1$, $i = 1, 2 \Longrightarrow \exists \{S_i^{n-1}\}_i, \{S_j^{n-1}\}_j$, that are respectively generated by the sets X_1 and X_2. So, we can take out from Σ_1^n two vacuum of dimension $n - 1$. We denote them by $\overset{\circ}{S}{}_{11}^{n-1}$ and $\overset{\circ}{S}{}_{12}^{n-1}$ that are respectively suitable to the simplexes S_{11}^{n-1} and S_{12}^{n-1}. Isomorphicly we take out S_{21}^n and S_{22}^n from Σ_2^n. We "stick" these borders to the isomorphic images from Σ_2^n. So we get the variety V_2^n. This is called the **variety of gender two**. We get inductively the countable set of varieties of respective gender:

$$V_0^n, V_1^n, V_2^n, \ldots, V_p^n, \ldots \tag{1}$$

The construction of set (1) was done by the countable set of finite sets $\{X_i\}_{i=1,\ldots,\infty}$, where $X_i \in F, \forall i \geq 1$. The set (1) satisfies the relation $\chi(V_p^n) = 2 - 2p$, $\forall p \geq 0$.

Let $F' = F \backslash \bigcup\limits_{i=1}^{\infty} X_i$ be a set of unused sets for (1). Similarly we construct one set of abstract, oriented varieties of odd dimension. In this case the varieties will be generated by the sets from F'. This set of varieties satisfies the relation $\chi(V_q^n) = 0$, $\forall q \geq 0$. So, now, the Euler characteristic cannot be used as a criterion of classification.

The set of all Δ-cycle of dimension m [3] of the variety V^n, $m = 0, 1, \ldots, n$, with respect to the addition of Δ-chains form a commutative group $Z^m(\Delta)$. There are two kind of Δ-cycles of V^n:

(a) $\Delta l^m = \Delta z^m = 0$;

(b) $\Delta\Delta l^m = \Delta(\Delta l^m) = 0$; $\Delta l^m \neq 0$.

The set of cycles of dimensions m with the property (a) forms a commutative group $Z_0^m(\Delta) \in Z^m(\Delta)$, $m = 0, 1, \ldots, n$. Let $r_0, r_1, \ldots, r_n$ be the ranks of the groups of Δ-homologies of the variety V^n, $Z^m(\Delta)/Z_0^m(\Delta) = \Delta^m(V^n, Z)$, $m = 1, 2, \ldots, n$. It is known [3] that

$$\chi(V_j^n) = \sum_{i=0}^{n} (-1)^i r_i,\ j \geq 0. \tag{2}$$

So, we can classify the abstract, oriented varieties with odd dimension by comparing the sequences $(r_0^j, r_1^j, \ldots, r_n^j)$, $j \geq 0$.

Definition 7. *If for the abstract, odd dimension varieties and without borders V_1^n and V_2^n the groups $\Delta^q(V_1^n, Z)$ and $\Delta^q(V_2^n, Z)$ are isomorphic, then V_1^n and V_2^n belong to the same class.*

This classification establishes the set of oriented varieties of odd dimension:

$$V_0^n, V_1^n, V_2^n, \ldots, V_q^n, \ldots \tag{3}$$

Theorem 8. *Let V^n be an arbitrary, abstract, oriented variety. There is one and only one element in* (1) *or* (3)*, V_p^n and V_q^n, so that $\chi(V^n) = \chi(V_p^n)$ or $\chi(V^n) = \chi(V_q^n)$.*

Similarly it is constructed the set of abstract, nonoriented varieties:

$$V_1^n, V_2^n, \ldots, V_l^n, \ldots \tag{4}$$

where $n \geq 2$ is even and $\chi(V_l^n) = 2 - l$.

Theorem 9. *Let V^n be an abstract, nonoriented variety, $n = 2m - 1 > 2$. There is one and only one element in* (4)*, V_l^n, so that $\chi(V^n) = \chi(V_l^n)$.*

So, the classification of abstract varieties of dimension n is done by there genders (modulo Euler characteristic).

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