REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 33 (2004) no. 2, pp. 167–174
ictp.acad.ro/jnaat

SOME PROCEDURES FOR SOLVING SPECIAL MAX-MIN FRACTIONAL RANK-TWO REVERSE-CONVEX PROGRAMMING PROBLEMS

DOINA IONAC* and ŞTEFAN ŢIGAN†

Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday

Abstract. In this paper, we suggest some procedures for solving two special classes of max-min fractional reverse-convex programs. We show that a special bilinear fractional max-min reverse-convex program can be solved by a linear reverse-convex programming problem.

For a linear fractional max-min reverse-convex program, possessing two reverse-convex sets, we propose a parametrical method. The particularity of this procedure is the fact that the max-min optimal solution of the original problem is obtained by solving at each iteration two linear reverse-convex programs with a rank-two monotonicity property.

MSC 2000. 26B25.
Keywords. Reverse-convex programming, max-min programming, bilinear fractional programs, linear fractional max-min programs.

1. INTRODUCTION

In this paper, we propose some methods for solving two special classes of max-min fractional reverse-convex programs.

Let $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ be given convex sets. Let T be a reverse-convex set in $\mathbb{R}^n$ (i.e. the complement of a convex set in $\mathbb{R}^n$) and S be a reverse-convex set in $\mathbb{R}^m$ and $h : X \times Y \to \mathbb{R}$. The general max-min reverse-convex programming consists in finding an optimal max-min solution for:

$$\textbf{PRC}. \qquad \max_{x\in X\cap T} \quad \min_{y\in Y\cap S} h(x,y).$$

We recall (see, e.g. [14, 17]) that a point $(x', y') \in (X \cap T) \times Y$ is said to be an *optimal* max-min *solution* for **PRC** problem, if the following two conditions hold:

$$(i) \qquad h(x',y') = \max_{x\in X\cap T} \quad \min_{y\in Y\cap S} h(x,y),$$
$$(ii) \qquad \min_{y\in Y\cap S} h(x',y) = h(x',y').$$

*Department of Mathematics, University of Oradea, e-mail: dionac@uoradea.ro.
†University of Medicine and Pharmacy "Iuliu-Haţieganu" Cluj-Napoca, Department of Medical Informatics and Biostatistics, e-mail: stigan@umfcluj.ro.

In the particular case, when **PRC** is a simple maximization problem only (i.e. Y has a single element, X is a polyhedral set and T is a reverse-convex set defined by a convex or quasi-convex constraint which has a rank-two monotonicity property), we recall that procedures for solving the maximum (or minimum) reverse-convex programming problems were given, for instance, in the papers [4], [7], [8], [10], [18], [6]. Duality aspects of the reverse-convex programming can be found in the paper by Penot [9]. For the max-min reverse-convex programming we mention the refs. [5], [6].

In section 2, for a special fractional type of the objective function h and for some particular cases of the sets X, Y, T and S, we obtain a particular case of problem **PRC** of bilinear fractional form, for which we propose, in Section 3, a method of finding optimal max-min solutions by reducing these problems to the particular case when X is a polyhedral set and T is a reverse-convex set defined by a convex or quasi-convex constraint that has a rank-two monotonicity property. These auxiliary problems can be solved by Kuno-Yamamoto procedure [8].

In section 4, we consider a max-min linear fractional reverse-convex programming problem, having two reverse-convex sets T and S, and for which we propose a parametric procedure.

Some concluding remarks are made in the last section.

2. BILINEAR FRACTIONAL MAX-MIN REVERSE-CONVEX PROGRAM PFM

Next we consider the following bilinear fractional max-min reverse-convex program (see, [6]) in which the reverse set $S = \mathbb{R}^m$:

PFM. Find

$$V = \max_{x\in X\cap T} \min_{y\in Y} \left(\frac{y^{\mathrm{T}}Cx + c^{\mathrm{T}}x + e^{\mathrm{T}}y + e_0}{\max\{\alpha_i^{\mathrm{T}} y + \beta_i | i \in I\}} \right) \tag{1}$$

where $I = \{1, 2, ..., r\}, C \in \mathbb{R}^{m\times n}, c \in \mathbb{R}^n, e, \alpha_i \in \mathbb{R}^m$ and $e_0, \beta_i \in \mathbb{R}$, $(i \in I)$.

The sets X and Y are defined by

$$X = \{x \in \mathbb{R}^n | Ax = a, x \geq 0\}, \tag{2}$$

$$Y = \{y \in \mathbb{R}^m | Dy \geq d, y \geq 0\}, \tag{3}$$

where $A \in \mathbb{R}^{s\times n}, a \in \mathbb{R}^s, D \in \mathbb{R}^{q\times m}, d \in \mathbb{R}^q$.

The reverse-convex set

$$T = \{x \in X^0 | f(x) \leq 0\}, \tag{4}$$

is defined by a function $f : \mathbb{R}^n \to \mathbb{R}$, which is continuous, strictly quasiconcave and has rank-two monotonicity on an open convex set $X^0 \subseteq \mathbb{R}^n$, which includes the set X.

We recall that f is said to be strictly quasiconcave on X^0 if for each $x, y \in X^0$ with $f(x) \neq f(y)$ we have $f((1 - t)x + ty) > \min\{f(x), f(y)\}$, for any $t \in (0, 1)$.

Definition 1. [8] *The function f possess a rank-two monotonicity on X^0 with respect to linearly independent vectors $\lambda_1, \lambda_2 \in \mathbb{R}^n$, if for any points $x', x'' \in X^0$, the inequality $\lambda_i x' \leq \lambda_i x''$, $i = 1, 2$ implies $f(x') \leq f(x'')$.*

We have the following representation for a function f possessing a rank-two monotonicity on X^0 with respect to linearly independent vectors $\lambda_1, \lambda_2 \in \mathbb{R}^n$.

Lemma 2. [8] *If the function f possess a rank-two monotonicity on X^0 with respect to linearly independent vectors $\lambda_1, \lambda_2 \in \mathbb{R}^n$, then there exists a function $g : \mathbb{R}^2 \to \mathbb{R}$, which is continuous and strictly quasiconcave on $\Gamma^0 = \{(\lambda_1 x, \lambda_2 x) | x \in X^0\}$ and satisfies the following two conditions:*
(i) *$f(x) = g(\lambda_1 x, \lambda_2 x)$, for $x \in X^0$,*
(ii) *$g(\theta) \leq g(\eta)$ if $\theta, \eta \in \Gamma^0$ and $\theta \leq \eta$.*

Concerning the problem **PFM**, we make the following assumptions:
A1. Y is a bounded non-empty set,
A2. the function f is continuous, strictly quasi-concave and has rank-two monotonicity on the open convex set X^0,
A3. $\max\{\alpha_i y + \beta_i | i \in I\} > 0$, $\forall y \in Y$.

3. LINEAR REVERSE-CONVEX PROGRAMMING APPROACH

We proposed in [6] a procedure for solving problem **PFM** based on the Charnes-Cooper [1] variable change.

Thus, if we perform in the problem (1)–(4), the Charnes-Cooper variable change $v = ty$, $t \geq 0$, $v \in \mathbb{R}^s$, (see, also Schaible [11], Stancu-Minasian [12] and Tigan [14], [16]) it follows by assumptions A1 and A3 that problem **PFM** is equivalent with:

PA. Find

$$V' = \max_{x \in X \cap T} \min_{(v,t) \in Y'} \left(v^{\mathrm{T}} C x + c^{\mathrm{T}} x t + e^{\mathrm{T}} v + e_0 t \right), \tag{5}$$

where the set Y' is defined by:

$$Dv - dt \geq 0, \tag{6}$$

$$\max\{\alpha_i^{\mathrm{T}} v + \beta_i t | i \in I\} \geq 1, \tag{7}$$

$$v \geq 0, t \geq 0. \tag{8}$$

We can show, by assumptions A1 and A3, that $V = V'$ and that for any optimal max-min solution (x^*, y^*) of problem **PFM** there exists an optimal max-min solution (x^*, v^*, t^*) of problem **PA** such that $y^* = \frac{v^*}{t^*}$ and conversely.

By Golstein [3] (see, also Ţigan and Stancu-Minasian [17]), the constraint (7) in Problem **PA** can be rewritten as the following maximum bilinear constraint:

$$\max_{\theta \in Z} \ \sum_{i=1}^{r} \theta_i (\alpha_i v + \beta_i t - 1) \geq 0, \tag{9}$$

where $Z = \{\theta \in \mathbb{R}^r | \sum_{i=1}^{r} \theta_i = 1, \theta_i \geq 0, i = 1, 2, ..., r\}$.

Let us denote

$$\psi(v,t) = \max_{\theta \in Z} \quad \sum_{i=1}^{r} \theta_i(\alpha_i^{\mathrm{T}} v + \beta_i t - 1), \forall (v,t) \in Y'. \tag{10}$$

By linear programming duality, for any $(v,t) \in Y'$, we have

$$\psi(v,t) = \min w \tag{11}$$

subject to

$$w \geq \alpha_i^{\mathrm{T}} v + \beta_i t - 1, \forall i \in I, w \geq 0. \tag{12}$$

From (10)–(12), it follows that the inequation (9) can be expressed by the following system:

$$w \geq 0,$$
$$w \geq \alpha_i^{\mathrm{T}} v + \beta_i t - 1, \forall i \in I.$$

Therefore, problem **PA** can be reduced to the following max-min bilinear reverse-convex program

PMM1. Find

$$V' = \max_{x \in X \cap T} \quad \min_{(v,t,w) \in Y''} \left(v^{\mathrm{T}}(Cx + e) + t(c^{\mathrm{T}} x + e_0) \right), \tag{13}$$

where the set Y'' is defined by:

$$Dv - dt \geq 0, \tag{14}$$

$$w \geq \alpha_i^{\mathrm{T}} v + \beta_i t - 1, \forall i \in I \tag{15}$$

$$v \geq 0, t \geq 0, w \geq 0. \tag{16}$$

From (13)–(16), by linear programming duality, for any $x \in X \cap T$, problem **PMM1** can be transformed into the following linear reverse-convex program:

PML. Find

$$V' = \max_{x,u,z} \left(-z_1 - z_2 - ... - z_r \right), \tag{17}$$

subject to

$$u^{\mathrm{T}} D - z^{\mathrm{T}} \Omega \leq Cx + e,$$
$$-u^{\mathrm{T}} d - z^{\mathrm{T}} \Lambda \leq c^{\mathrm{T}} x + e_0, \ i \in I$$
$$x \in X \cap T,$$
$$u \geq 0, z \geq 0, u \in \mathbb{R}^q, z \in \mathbb{R}^r.$$

In problem **PML**, we denote by $\Omega \in \mathbb{R}^{r \times s}$ the matrix having the rows α_i $(i \in I)$, and by Λ the vector $\Lambda = (\beta_1, ..., \beta_r)^{\mathrm{T}} \in \mathbb{R}^r$.

Therefore, we proved the following theorem:

THEOREM 3. *If the problem* **PFM** *satisfies the assumptions A1-A3, then problem* **PFM** *can be solved by solving the linear reverse-convex program with a rank-two monotonicity* **PML**.

In order to solve problem **PFM**, a procedure similar to Algorithm 1 can be used, by replacing in step1 the problem **PLC** by the linear reverse-convex programming problem **PML**.

ALGORITHM 1. *Step 1. Solve the linear reverse-convex programs with a rank-two monotonicity* **PML**.

If $V < \infty$ *and the feasible set* X' *of* **PML** *is non-empty, let* x^* *be the corresponding component of an optimal solution of problem* **PML**.

If $X = \emptyset$, *then take* $V = -\infty$.

Step 2. i) *If* $-\infty < V < \infty$, *then* (x^*, y^*) *is an optimal solution of* max-min *problem* **PFM**, *where* y^* *is an optimal solution of the generalized linear-fractional program* **PFA**.

ii) *If* $V = -\infty$, *then* **PFM** *is unfeasible.*

iii) *If* $V = \infty$, *then* **PFM** *has an infinite optimum.*

We make the remark that auxiliary linear reverse-convex programming problem **PML** in Algorithm 1 is simpler than the auxiliary linear reverse-convex programming problem in the algorithm proposed for this problem in [6]. Indeed, problem **PML** has only $s+r$ linear constraints the auxiliary problem in [6] posses $(s+2)r$ linear constraints. Therefore, Algorithm 1 seems to be more efficient than algorithm proposed for this problem in [6].

4. MAX-MIN LINEAR FRACTIONAL REVERSE-CONVEX PROGRAMMING WITH TWO SEPARATE REVERSE-CONVEX FEASIBLE SETS

In this section, we consider the following max-min linear fractional program **GLF**. Find

$$\max_{x\in X\cap T} \min_{y\in Y\cap S} h(x,y)$$

where

$$h(x,y) = \frac{\alpha^{\mathrm{T}}x + \beta^{\mathrm{T}}y + \beta_0}{\gamma^{\mathrm{T}}x + \eta^{\mathrm{T}}y + \eta_0},$$

verifying the condition

$$\gamma^{\mathrm{T}}x + \eta^{\mathrm{T}}y + \eta_0 > 0,\ \forall x \in X,\ \forall y \in Y.$$

In problem **GLF**, X and S are defined by

$$X = \{x \in \mathbb{R}^n | Ax = a, x \geq 0\}, \tag{18}$$

$$Y = \{y \in \mathbb{R}^m | By = b, y \geq 0\}, \tag{19}$$

where $A \in \mathbb{R}^{s\times n}, B \in \mathbb{R}^{p\times m}, a \in \mathbb{R}^s, b \in \mathbb{R}^p, \alpha, \gamma \in \mathbb{R}^n, \beta, \eta \in \mathbb{R}^m, \beta_0, \eta_0 \in \mathbb{R}$, are given matrices, vectors and real constants respectively.

The reverse-convex set

$$T = \{x \in X^0 | f(x) \leq 0\}, \tag{20}$$

is defined by a function $f : \mathbb{R}^n \to \mathbb{R}$, which is continuous, strictly quasiconcave and has rank-two monotonicity on an open convex set $X^0 \subseteq \mathbb{R}^n$, which includes the set X and the reverse-convex set

$$S = \{y \in Y^0 | f_1(y) \leq 0\}, \tag{21}$$

is defined by a function $f_1 : \mathbb{R}^m \to \mathbb{R}$, which is continuous, strictly quasiconcave and has rank-two monotonicity on an open convex set $Y^0 \subseteq \mathbb{R}^m$, which includes the set Y.

For solving problem **GLF** we can use a parametric procedure (see, [2], [13], [15]), by which an approximate optimal solution could be found by solving a sequence of the auxiliary reverse-convex programs each of them having only one reverse-convex constraint.

ALGORITHM 2. *Let $\varepsilon > 0$ be a given positive real number, representing a level of approximation to be attain by algorithm.*

1. Find a point $x^0 \in X \cap T$ and a point $y^0 \in Y \cap S$ and set $k := 0$.

2. Take

$$t_k = h(x^k, y^k).$$

3. Find

$$F(t_k) = \max_{x \in X \cap T} \min_{y \in Y \cap S} [(\alpha - t_k\gamma)x + (\beta - t_k\eta)y + \beta_0 - t_k\eta_0]. \tag{22}$$

But the max-min program (22) can be transformed into the following two linear reverse-convex programs

PL1. Find

$$\max_x (\alpha - t_k\gamma)x \tag{23}$$

subject to

$$x \in X \cap T. \tag{24}$$

PL2. Find

$$\min_y [(\beta - t_k\eta)y + \beta_0 - t_k\eta_0] \tag{25}$$

subject to

$$y \in Y \cap S. \tag{26}$$

Let x^{k+1}, y^{k+1} be an optimal solution of the linear reverse-convex program (23)–(24) and (25)–(26), respectively. Obviously, we have $F(t_k) = (\alpha - t_k\gamma)x^{k+1} + (\beta - t_k\eta)y^{k+1} + \beta_0 - t_k\eta_0$.

4. i) If $F(t_k) \leq \varepsilon$, then (x^{k+1}, y^{k+1}) is an approximate optimal solution of problem **GLF.**

ii) If $F(t_k) > \varepsilon$, then take k:=k+1 and go to the step 2.

5. CONCLUSIONS

In this paper we consider two fractional max-min reverse-convex programming problems.

Firstly, we give a new procedure for solving a particular class of max-min bilinear fractional reverse-convex programming problems. The particularity of this procedure is the fact that the max-min optimal solution of the original problem is obtained by solving a single linear reverse-convex program with a rank-two monotonicity with an algorithm proposed by Kuno and Yamamoto [8].

Secondly, we consider a parametric procedure for solving a particular class of max-min linear fractional reverse-convex programming problems, possessing two reverse-convex feasible sets. The particularity of this procedure is the fact that the max-min optimal solution of the original problem is obtained by solving at each iteration two linear reverse-convex programs with a rank-two monotonicity with the algorithm of Kuno and Yamamoto.

REFERENCES

[1] Charnes, A., Cooper, W.W., *Programming with linear fractional functionals*, Naval Res. Logist. Quart., **9**, 1–2, pp. 181–186, 1962.

[2] Crouzeix, J.P., Ferland, J.A. and Schaible, S., *An algorithm for generalized fractional programs*, J. Optim. Theory Appl., **47**, 1, pp. 35–49, 1985.

[3] Golstein, E.G., *Duality theory in mathematical programming and its applications*, Nauka, Moskva, 1971 (in Russian).

[4] Hillestad, R.J., Jacobsen S.E., *Linear programs with an additional reverse-convex constraint*, Applied Mathematics and Optimization, **6**, pp. 257–269, 1980.

[5] Ionac, D., *Aspecte privind analiza unor probleme de programare matematică*, ed. Treira, Oradea, 2000 (in Romanian).

[6] Ionac, D. and Tigan, S., *Solving procedures for some max-min reverse-convex programs*, Proc. of the "Tiberiu Popoviciu" Itinerant Seminar on functional Equations, Approximation and Convexity, Editura SRIMA, Cluj-Napoca, Romania, pp. 105–118, 2002.

[7] Kuno, T., *Globally determining a minimum area rectangle enclosing the projection of a higher-dimensional set*, Operations Research Letters, **13**, pp. 295–303, 1993.

[8] Kuno, T. and Yamamoto Y., *A finite algorithm for globally optimizing a class of rank-two reverse-convex programming*, Journal of Global Optimization, **12**, 3, pp. 247–265, 1998.

[9] Penot, J.-P., *Duality for anticonvex programs*, Journal of Global Optimization, **19**, pp. 163–182, 2001.

[10] Pferschy, U. and Tuy, H., *Linear programs with an additional rank-two reverse-convex constraint*, Journal of Global Optimization, **4**, pp. 441–454, 1994.

[11] Schaible, S., *Nonlinear fractional programming*, Oper. Res. Verfahren, **19**, pp. 109–115, 1974.

[12] Stancu-Minasian, I.M., *Fractional Programming Theory, Methods and Applications,* Kluwer Academic Publishers, Dordrecht, 1997.

[13] Tigan, S., *Sur une methode de resolution d'un problème d'optimisation par segments*, Rev. Anal. Numér. Théor. Approx., **4**, no. 1, pp. 87–97, 1975.

[14] Tigan, S., *On the* max-min *nonlinear fractional problem*, Rev. Anal. Numér. Théor. Approx., **9**, no. 2, pp. 283–288, 1980.

[15] TIGAN, S., *A parametrical method for max-min nonlinear fractional problems*, Proc. of the Itinerant Seminar on functional Equations, Approximation and Convexity, "Babeş-Bolyai" University, Cluj-Napoca, pp. 175–184, 1983.
[16] TIGAN, S., *Asupra unor probleme fracţionare de maximin*, Studii şi Cercetări de Calcul Economic şi Calcul Economic, 1–2, pp. 53–57, 1992.
[17] TIGAN, S. and STANCU-MINASIAN, I.M., *Methods for solving stochastic bilinear fractional max-min problems*, Recherche Opérationnelle / Operations Research, **30**, no. 1, pp. 81–98, 1996.
[18] TUY, H., *Convex programs with an additional reverse-convex constraint*, Journal of Optimization Theory and Applications, **52**, pp. 463–486, 1987.

Received by the editors: May 11, 2004.