SOME PROCEDURES FOR SOLVING SPECIAL MAX-MIN FRACTIONAL RANK-TWO REVERSE-CONVEX PROGRAMMING PROBLEMS

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday

Abstract. In this paper, we suggest some procedures for solving two special classes of max-min fractional reverse-convex programs. We show that a special bilinear fractional max-min reverse-convex program can be solved by a linear reverse-convex programming problem.

For a linear fractional max-min reverse-convex program, possessing two reverse-convex sets, we propose a parametrical method. The particularity of this procedure is the fact that the max-min optimal solution of the original problem is obtained by solving at each iteration two linear reverse-convex programs with a rank-two monotonicity property.


Keywords. Reverse-convex programming, max-min programming, bilinear fractional programs, linear fractional max-min programs.

1. INTRODUCTION

In this paper, we propose some methods for solving two special classes of max-min fractional reverse-convex programs.

Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ be given convex sets. Let $T$ be a reverse-convex set in $\mathbb{R}^n$ (i.e. the complement of a convex set in $\mathbb{R}^n$) and $S$ be a reverse-convex set in $\mathbb{R}^m$ and $h : X \times Y \to \mathbb{R}$. The general max-min reverse-convex programming consists in finding an optimal max-min solution for:

$$\text{PRC.} \quad \max_{x \in X \cap T} \min_{y \in Y \cap S} h(x, y).$$

We recall (see, e.g. [14, 17]) that a point $(x', y') \in (X \cap T) \times Y$ is said to be an optimal max-min solution for PRC problem, if the following two conditions hold:

1. $h(x', y') = \max_{x \in X \cap T} \min_{y \in Y \cap S} h(x, y),$
2. $\min_{y \in Y \cap S} h(x', y) = h(x', y').$

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In the particular case, when PRC is a simple maximization problem only (i.e. $Y$ has a single element, $X$ is a polyhedral set and $T$ is a reverse-convex set defined by a convex or quasi-convex constraint which has a rank-two monotonicity property), we recall that procedures for solving the maximum (or minimum) reverse-convex programming problems were given, for instance, in the papers [4], [7], [8], [10], [15], [6]. Duality aspects of the reverse-convex programming can be found in the paper by Penot [9]. For the max-min reverse-convex programming we mention the refs. [5], [6].

In section 2, for a special fractional type of the objective function $h$ and for some particular cases of the sets $X, Y, T$ and $S$, we obtain a particular case of problem PRC of bilinear fractional form, for which we propose, in Section 3, a method of finding optimal max-min solutions by reducing these problems to the particular case when $X$ is a polyhedral set and $T$ is a reverse-convex set defined by a convex or quasi-convex constraint that has a rank-two monotonicity property. These auxiliary problems can be solved by Kuno-Yamamoto procedure [8].

In section 4, we consider a max-min linear fractional reverse-convex programming problem, having two reverse-convex sets $T$ and $S$, and for which we propose a parametric procedure.

Some concluding remarks are made in the last section.

2. BILINEAR FRACTIONAL MAX-MIN REVERSE-CONVEX PROGRAM PFM

Next we consider the following bilinear fractional max-min reverse-convex program (see, [6]) in which the reverse set $S = \mathbb{R}^m$:

**PFM.** Find

\[
V = \max_{x \in X \cap T, y \in Y} \min \left( y^T C x + c^T x + e^T y + e_0 \middle| \max \{\alpha_i^T y + \beta_i | i \in I\} \right)
\]

where $I = \{1, 2, ..., r\}$, $C \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $e, \alpha_i \in \mathbb{R}^m$ and $e_0, \beta_i \in \mathbb{R}$, $(i \in I)$.

The sets $X$ and $Y$ are defined by

\[
X = \{x \in \mathbb{R}^n | Ax = a, x \geq 0\},
\]

\[
Y = \{y \in \mathbb{R}^m | Dy \geq d, y \geq 0\},
\]

where $A \in \mathbb{R}^{s \times n}$, $a \in \mathbb{R}^s$, $D \in \mathbb{R}^{q \times m}$, $d \in \mathbb{R}^q$.

The reverse-convex set

\[
T = \{x \in X^0 | f(x) \leq 0\},
\]

is defined by a function $f : \mathbb{R}^n \to \mathbb{R}$, which is continuous, strictly quasiconcave and has rank-two monotonicity on an open convex set $X^0 \subseteq \mathbb{R}^n$, which includes the set $X$.

We recall that $f$ is said to be strictly quasiconcave on $X^0$ if for each $x, y \in X^0$ with $f(x) \neq f(y)$ we have $f((1 - t)x + ty) > \min\{f(x), f(y)\}$, for any $t \in (0, 1)$. 
**Definition 1.** [8] The function $f$ possess a rank-two monotonicity on $X^0$ with respect to linearly independent vectors $\lambda_1, \lambda_2 \in \mathbb{R}^n$, if for any points $x', x'' \in X^0$, the inequality $\lambda_i x' \leq \lambda_i x''$, $i = 1, 2$ implies $f(x') \leq f(x'')$.

We have the following representation for a function $f$ possessing a rank-two monotonicity on $X^0$ with respect to linearly independent vectors $\lambda_1, \lambda_2 \in \mathbb{R}^n$.

**Lemma 2.** [8] If the function $f$ possess a rank-two monotonicity on $X^0$ with respect to linearly independent vectors $\lambda_1, \lambda_2 \in \mathbb{R}^n$, then there exists a function $g : \mathbb{R}^2 \to \mathbb{R}$, which is continuous and strictly quasiconcave on $\Gamma^0 = \{(\lambda_1 x, \lambda_2 x) | x \in X^0\}$ and satisfies the following two conditions:

(i) $f(x) = g(\lambda_1 x, \lambda_2 x)$, for $x \in X^0$,

(ii) $g(\theta) \leq g(\eta)$ if $\theta, \eta \in \Gamma^0$ and $\theta \leq \eta$.

Concerning the problem **PFM**, we make the following assumptions:

A1. $Y$ is a bounded non-empty set,

A2. the function $f$ is continuous, strictly quasi-concave and has rank-two monotonicity on the open convex set $X^0$,

A3. $\max\{\alpha_i y + \beta_i | i \in I\} > 0$, $\forall y \in Y$.

### 3. Linear Reverse-Convex Programming Approach

We proposed in [6] a procedure for solving problem **PFM** based on the Charnes-Cooper [1] variable change. Thus, if we perform in the problem (1)–(4), the Charnes-Cooper variable change $v = ty$, $t \geq 0$, $v \in \mathbb{R}^s$, (see, also Schaible [11], Stancu-Minasian [12] and Tigan [14], [16]) it follows by assumptions A1 and A3 that problem **PFM** is equivalent with:

**PA.** Find

$$V' = \max_{x \in X \cap T(v,t) \in Y'} \min \left( v^T C x + \epsilon^T x + \epsilon_0 t \right),$$

where the set $Y'$ is defined by:

$$Dv - dt \geq 0,$$

$$\max\{\alpha_i t + \beta_i t | i \in I\} \geq 1,$$

$$v \geq 0, t \geq 0.$$

We can show, by assumptions A1 and A3, that $V = V'$ and that for any optimal max-min solution $(x^*, y^*)$ of problem **PFM** there exists an optimal max-min solution $(x^*, v^*, t^*)$ of problem **PA** such that $y^* = \frac{v^*}{t^*}$ and conversely.

By Golstein [3] (see, also Tigan and Stancu-Minasian [17]), the constraint (7) in Problem **PA** can be rewritten as the following maximum bilinear constraint:

$$\max_{\theta \in \mathbb{Z}} \sum_{i=1}^{r} \theta_i (\alpha_i v + \beta_i t - 1) \geq 0,$$
where \( Z = \{ \theta \in \mathbb{R}^r \mid \sum_{i=1}^{r} \theta_i = 1, \theta_i \geq 0, i = 1, 2, ..., r \} \).

Let us denote

\[
\psi(v,t) = \max_{\theta \in Z} \sum_{i=1}^{r} \theta_i (\alpha_i^T v + \beta_i t - 1), \forall (v,t) \in Y'.
\]

By linear programming duality, for any \((v,t) \in Y'\), we have

\[
\psi(v,t) = \min w
\]

subject to

\[
w \geq \alpha_i^T v + \beta_i t - 1, \forall i \in I, w \geq 0.
\]

From (10)–(12), it follows that the inequation (9) can be expressed by the following system:

\[
w \geq 0,
\]

\[
w \geq \alpha_i^T v + \beta_i t - 1, \forall i \in I.
\]

Therefore, problem \( PA \) can be reduced to the following max-min bilinear reverse-convex program

\(\text{PMM1} \).

\[
V' = \max_{x \in X \cap T} \min_{(v,t,w) \in Y''} \left( v^T (C x + e) + t (c^T x + e_0) \right),
\]

where the set \( Y'' \) is defined by:

\[
Dv - dt \geq 0,
\]

\[
w \geq \alpha_i^T v + \beta_i t - 1, \forall i \in I
\]

\[
v \geq 0, t \geq 0, w \geq 0.
\]

From (13)–(16), by linear programming duality, for any \( x \in X \cap T \), problem \( \text{PMM1} \) can be transformed into the following linear reverse-convex program:

\(\text{PML} \).

\[
V' = \max_{x,u,z} (-z_1 - z_2 - ... - z_r),
\]

subject to

\[
u^T D - z^T \Omega \leq C x + e,
\]

\[
-u^T d - z^T \Lambda \leq c^T x + e_0, i \in I
\]

\[
x \in X \cap T,
\]

\[
u \geq 0, z \geq 0, u \in \mathbb{R}^q, z \in \mathbb{R}^r.
\]

In problem \(\text{PML} \), we denote by \( \Omega \in \mathbb{R}^{r \times s} \) the matrix having the rows \( \alpha_i \) \((i \in I)\), and by \( \Lambda \) the vector \( \Lambda = (\beta_1, ..., \beta_r)^T \in \mathbb{R}^r \).

Therefore, we proved the following theorem:
THEOREM 3. If the problem PFM satisfies the assumptions A1-A3, then problem PFM can be solved by solving the linear reverse-convex program with a rank-two monotonicity PML.

In order to solve problem PFM, a procedure similar to Algorithm 1 can be used, by replacing in step 1 the problem PLC by the linear reverse-convex programming problem PML.

Algorithm 1. Step 1. Solve the linear reverse-convex programs with a rank-two monotonicity PML.

If \( V < \infty \) and the feasible set \( X' \) of PML is non-empty, let \( x^* \) be the corresponding component of an optimal solution of problem PML.

If \( X = \emptyset \), then take \( V = -\infty \).

Step 2. i) If \( -\infty < V < \infty \), then \((x^*, y^*)\) is an optimal solution of max-min problem PFM, where \( y^* \) is an optimal solution of the generalized linear-fractional program PFA.

ii) If \( V = -\infty \), then PFM is unfeasible.

iii) If \( V = \infty \), then PFM has an infinite optimum.

We make the remark that auxiliary linear reverse-convex programming problem PML in Algorithm 1 is simpler than the auxiliary linear reverse-convex programming problem in the algorithm proposed for this problem in [6]. Indeed, problem PML has only \( s + r \) linear constraints the auxiliary problem in [6] posses \((s + 2)r\) linear constraints. Therefore, Algorithm 1 seems to be more efficient than algorithm proposed for this problem in [6].

4. MAX-MIN LINEAR FRACTIONAL REVERSE-CONVEX PROGRAMMING WITH TWO SEPARATE REVERSE-CONVEX FEASIBLE SETS

In this section, we consider the following max-min linear fractional program GLF. Find

\[
\max_{x \in X \cap T} \min_{y \in Y \cap S} h(x, y)
\]

where

\[
h(x, y) = \frac{\alpha^T x + \beta^T y + \beta_0}{\gamma^T x + \eta^T y + \eta_0},
\]

verifying the condition

\[
\gamma^T x + \eta^T y + \eta_0 > 0, \forall x \in X, \forall y \in Y.
\]

In problem GLF, \( X \) and \( S \) are defined by

\[
X = \{x \in \mathbb{R}^n | Ax = a, x \geq 0\},
\]

\[
Y = \{y \in \mathbb{R}^m | By = b, y \geq 0\},
\]

where \( A \in \mathbb{R}^{s \times n}, B \in \mathbb{R}^{p \times m}, a \in \mathbb{R}^s, b \in \mathbb{R}^p, \alpha \in \mathbb{R}^n, \beta, \eta \in \mathbb{R}^m, \beta_0, \eta_0 \in \mathbb{R}, \) are given matrices, vectors and real constants respectively.
The reverse-convex set

\[(20) \quad T = \{ x \in X^0 | f(x) \leq 0 \}, \]

is defined by a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), which is continuous, strictly quasiconcave and has rank-two monotonicity on an open convex set \( X^0 \subseteq \mathbb{R}^n \), which includes the set \( X \) and the reverse-convex set

\[(21) \quad S = \{ y \in Y^0 | f_1(y) \leq 0 \}, \]

is defined by a function \( f_1 : \mathbb{R}^m \rightarrow \mathbb{R} \), which is continuous, strictly quasiconcave and has rank-two monotonicity on an open convex set \( Y^0 \subseteq \mathbb{R}^m \), which includes the set \( Y \).

For solving problem \( \text{GLF} \) we can use a parametric procedure (see, [2], [13], [15]), by which an approximate optimal solution could be found by solving a sequence of the auxiliary reverse-convex programs each of them having only one reverse-convex constraint.

**Algorithm 2.** Let \( \varepsilon > 0 \) be a given positive real number, representing a level of approximation to be attain by algorithm.

1. Find a point \( x^0 \in X \cap T \) and a point \( y^0 \in Y \cap S \) and set \( k := 0 \).
2. Take

\[ t_k = h(x^k, y^k). \]

3. Find

\[(22) \quad F(t_k) = \max_{x \in X \cap T} \min_{y \in Y \cap S} [(\alpha - t_k \gamma)x + (\beta - t_k \eta)y + \beta_0 - t_k \eta_0]. \]

But the max-min program \( (22) \) can be transformed into the following two linear reverse-convex programs

**PL1.** Find

\[(23) \quad \max_x (\alpha - t_k \gamma)x \]

subject to

\[(24) \quad x \in X \cap T. \]

**PL2.** Find

\[(25) \quad \min_y [(\beta - t_k \eta)y + \beta_0 - t_k \eta_0] \]

subject to

\[(26) \quad y \in Y \cap S. \]

Let \( x^{k+1}, y^{k+1} \) be an optimal solution of the linear reverse-convex program \( (23)-(24) \) and \( (25)-(26) \), respectively. Obviously, we have \( F(t_k) = (\alpha - t_k \gamma)x^{k+1} + (\beta - t_k \eta)y^{k+1} + \beta_0 - t_k \eta_0 \).

4. i) If \( F(t_k) \leq \varepsilon \), then \( (x^{k+1}, y^{k+1}) \) is an approximate optimal solution of problem \( \text{GLF} \).
   
   ii) If \( F(t_k) > \varepsilon \), then take \( k := k+1 \) and go to the step 2.
5. CONCLUSIONS

In this paper we consider two fractional max-min reverse-convex programming problems.

Firstly, we give a new procedure for solving a particular class of max-min bilinear fractional reverse-convex programming problems. The particularity of this procedure is the fact that the max-min optimal solution of the original problem is obtained by solving a single linear reverse-convex program with a rank-two monotonicity with an algorithm proposed by Kuno and Yamamoto [8].

Secondly, we consider a parametric procedure for solving a particular class of max-min linear fractional reverse-convex programming problems, possessing two reverse-convex feasible sets. The particularity of this procedure is the fact that the max-min optimal solution of the original problem is obtained by solving at each iteration two linear reverse-convex programs with a rank-two monotonicity with the algorithm of Kuno and Yamamoto.

REFERENCES


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