FINITELY DEFINED FUNCTIONALS AND DIVIDED DIFFERENCES

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. We give a necessary and sufficient condition for representing finitely defined functionals in terms of divided differences. As particular cases we obtain formulas of Tiberiu Popoviciu, Newton, etc.

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1. PRELIMINARIES AND NOTATIONS

Let $n$ be a positive integer. We use the following notations andabbreviations:

• $\mathcal{P}_n$, the linear space of polynomials of degree at most $n$;
• $e_i \in \mathcal{P}_n$, $e_i(t) = t^i$, $i = 0, \ldots, n$;
• $x_0, \ldots, x_n$, distinct points of an interval $[a, b]$;
• $\mathcal{F}$, the linear space of all real functions defined on $\{x_0, \ldots, x_n\}$;
• $\mathcal{A}_n$, a set of linear finitely defined functionals,

$$\mathcal{A}_n = \left\{ A : \mathcal{F} \to \mathbb{R} \mid A(f) = \sum_{i=0}^{n} a_i f(x_i), \ a_0, \ldots, a_n \in \mathbb{R} \right\};$$

• $[t_1, \ldots, t_n; f]$, the divided difference of the function $f$, with respect to the distinct nodes $t_1, \ldots, t_n$;
• $\{x_{i,0}, \ldots, x_{i,n_i}\}, i = 0, \ldots, n$, nonempty subsets of $\{x_0, \ldots, x_n\}$;
• $u_+ := \begin{cases} 0, & \text{if } u \leq 0; \\ u, & \text{if } u > 0, \end{cases}$ the positive part of $u$;

**Lemma 1.** If $f_0, \ldots, f_n \in \mathcal{F}$ are linearly independent and $F_0, \ldots, F_n \in \mathcal{A}_n$ satisfy the condition

$$d := \det(F_i(f_j))_{i,j=0}^{n} \neq 0,$$
then, for any $A \in \mathcal{A}_n$, the following formula is satisfied

\[ A = \sum_{k=0}^{n} \frac{(-1)^k}{d} \begin{vmatrix} A(f_0) & \cdots & A(f_n) \\ F_0(f_0) & \cdots & F_0(f_n) \\ \vdots & \ddots & \vdots \\ F_k(f_0) & \cdots & F_k(f_n) \\ F_n(f_0) & \cdots & F_n(f_n) \end{vmatrix} \cdot F_k \]  \tag{2}

(the notation $\not\!F_k$ means that the $k$-th row is canceled).

**Proof.** Let $f \in \mathcal{F}$. Taking into account the fact that $f$ is a linear combination of $f_0, \ldots, f_n$, it follows that

\[
\begin{vmatrix} A(f_0) & \cdots & A(f_n) \\ F_0(f_0) & \cdots & F_0(f_n) \\ \vdots & \ddots & \vdots \\ F_n(f_0) & \cdots & F_n(f_n) \end{vmatrix} = 0.
\]

We expand the determinant in terms of the first column and we take into consideration (1). □

Consider the polynomials

\[ P_i(t) := \frac{\ell(t)}{(t - x_{i,0}) \ldots (t - x_{i,n})}, \]

$i = 0, \ldots, n$, where $\ell(t) := (t - x_0) \ldots (t - x_n)$.

For $y_0, \ldots, y_p \in \{x_0, \ldots, x_n\}$, the reduction formula for divided differences gives

\[ \left[ x_0, \ldots, x_n; \frac{\ell(t)}{(t - y_0) \ldots (t - y_p)} f(t) \right]_t = [y_0, \ldots, y_p; f]. \]

The the polynomials $P_i$ satisfy

\[ [x_0, \ldots, x_n; P_i \cdot f] = [x_{i,0} \ldots, x_{i,n}; f]. \]  \tag{4}

**Lemma 2.** If $Q \in \mathcal{P}_n$ and

\[ [x_0, \ldots, x_n; P Q] = 0, \quad \forall P \in \mathcal{P}_n, \]

then $Q = 0$.

**Proof.** By virtue of (1), with $P_i(t) = \ell(t)/(t - x_i), i = 0, \ldots, n$, we obtain

\[ Q(x_i) = [x_i; Q] = [x_0, \ldots, x_n; P_i \cdot Q] = 0, \quad i = 0, \ldots, n, \]

hence $Q = 0$. □
2. MAIN RESULTS

**Theorem 3.** Any functional \( A \in A_n \) can be written in the form

\[
A = \sum_{k=0}^{n} \lambda_k [x_{k,0}, \ldots, x_{k,n}; \cdot],
\]

for some \( \lambda_0, \ldots, \lambda_n \in \mathbb{R} \), if and only if the polynomials \( P_0, \ldots, P_n \) are linearly independent.

**Proof. Necessity.** Let \( P \in P_n \) be an arbitrary polynomial of degree \( n \). Consider the linear functional

\[
A(f) = [x_0, \ldots, x_n; P \cdot f].
\]

It follows that there exist \( \lambda_k \in \mathbb{R}, k = 0, \ldots, n \), such that

\[
A(f) = \sum_{k=0}^{n} \lambda_k [x_{k,0}, \ldots, x_{k,n}; f], \quad \forall f \in F.
\]

By using (4), we obtain

\[
A(f) = \sum_{k=0}^{n} \lambda_k [x_0, \ldots, x_n; P_k \cdot f] = \sum_{k=0}^{n} \lambda_k P_k \cdot f],
\]

Consequently,

\[
\left[ x_0, \ldots, x_n; \left( P - \sum_{k=0}^{n} \lambda_k P_k \right) f \right] = 0, \quad \forall f \in F,
\]

therefore, by Lemma 2, we obtain

\[
P = \sum_{k=0}^{n} \lambda_k P_k,
\]

that is the set \( \{P_0, \ldots, P_n\} \) is a basis in \( P_n \). It follows that the polynomials \( P_0, \ldots, P_n \) are linearly independent.

**Sufficiency.** Suppose that the polynomials \( P_0, \ldots, P_n \) are linearly independent and let \( A \in A_n, A(f) = \sum_{k=0}^{n} a_k f(x_k) \). There exist \( \lambda_0, \ldots, \lambda_n \in \mathbb{R} \) such that

\[
\sum_{k=0}^{n} a_k \frac{f(t)}{t-x_k} = \sum_{k=0}^{n} \lambda_k P_k(t), \quad t \in \{x_0, \ldots, x_n\}.
\]

It follows that

\[
\sum_{k=0}^{n} a_k \frac{f(t)}{t-x_k} f(t) = \sum_{k=0}^{n} \lambda_k P_k(t) f(t), \quad t \in \{x_0, \ldots, x_n\},
\]

hence

\[
\left[ x_0, \ldots, x_n; \sum_{k=0}^{n} a_k \frac{f(t)}{t-x_k} f(t) \right]_t = \left[ x_0, \ldots, x_n; \sum_{k=0}^{n} \lambda_k P_k(t) f(t) \right]_t.
\]
therefore
\[ \sum_{k=0}^{n} \left[ x_0, \ldots, x_n; a_k \ell(t) f(t) \right] = \sum_{k=0}^{n} \left[ x_0, \ldots, x_n; \lambda_k P_k(t) f(t) \right]. \]
Consequently, by using (4), it follows that
\[ \sum_{k=0}^{n} a_k f(x_k) = \sum_{k=0}^{n} \lambda_k [x_{k,0}, \ldots, x_{k,n}; f]; \]
that is,
\[ A(f) = \sum_{k=0}^{n} \lambda_k [x_{k,0}, \ldots, x_{k,n}; f]. \]

In what follows we consider the functionals (6)
\[ D_k := [x_{k,0}, \ldots, x_{k,n}; \cdot], \quad k = 0, \ldots, n. \]

**Lemma 4.** If \( \det(P_i(x_j))_{i,j=0}^{n} \neq 0 \), then \( \delta := \det(D_k(P_i))_{i,k=0}^{n} \neq 0. \)

**Proof.** We consider the functionals \( A_j \in \mathcal{A}_n \),
\[ A_j(f) = f(x_j), \quad j = 0, \ldots, n. \]
Since the polynomials \( \{P_i\} \) are linearly independent, by virtue of Theorem 3, it follows that there exists numbers \( \lambda_{jk} \) such that
\[ A_j = \sum_{k=0}^{n} \lambda_{jk} D_k, \]
hence
\[ P_i(x_j) = A_j(P_i) = \sum_{k=0}^{n} \lambda_{jk} D_k(P_i), \]
i, j = 0, \ldots, n. We get
\[ \det(P_i(x_j))_{i,j=0}^{n} = \det(\lambda_{jk})_{k,j=0}^{n} \cdot \det(D_k(P_i))_{i,k=0}^{n}, \]
therefore \( \det(D_k(P_i))_{i,k=0}^{n} \neq 0 \).

In the next theorem we give a representation of \( \lambda_k \) from Eq. (5) in terms of the functionals (6).

**Theorem 5.** If \( \det(P_i(x_j))_{i,j=0}^{n} \neq 0 \) and \( A \in \mathcal{A}_n \), then
\[ A = \sum_{k=0}^{n} \left( -1 \right)^k \frac{1}{\delta} \begin{vmatrix} A(P_0) & \cdots & A(P_n) \\ D_0(P_0) & \cdots & D_0(P_n) \\ \vdots & \ddots & \vdots \\ \Psi_k(P_0) /\Psi_k(P_n) & \cdots & \Psi_k(P_0)/\Psi_k(P_n) \\ \vdots & \ddots & \vdots \\ D_n(P_0) & \cdots & D_n(P_n) \end{vmatrix} [x_{k,0}, \ldots, x_{k,n}; \cdot]. \]
Proof. We use Theorem 3, Lemma 1 and Lemma 4.

3. APPLICATIONS

Corollary 6 (Popoviciu’s Transformation Formula [3 Eq. (17)]). If \( r \) is an integer, \( 0 \leq r \leq n \), then for all \( A \in \mathcal{A}_n \) there exist numbers \( \alpha_i \) such that

\[
A(f) = \sum_{0 \leq i \leq r-1} \alpha_i [x_0, \ldots, x_i; f] + \sum_{r \leq i \leq n} \alpha_i [x_{i-r}, \ldots, x_i; f].
\]

Proof. In Theorem 3, we take

\[
\mathcal{P}_i(t) := \begin{cases} \frac{\ell(t)}{(t-x_0)(t-x_1)}, & 0 \leq i \leq r-1, \\ \frac{\ell(t)}{(t-x_{i-r})(t-x_i)}, & r \leq i \leq n. \end{cases}
\]

Corollary 7. [2]. If \( a = x_0 < \cdots < x_n = b \) and \( U : \mathcal{F}[a, b] \to \mathcal{F}[a, b] \) is a linear operator, then

\[
U(f; x) = U(e_0; x) f(x_0) + U(e_1 - x_0 e_0; x) [x_0, x_1; f] + \sum_{k=0}^{n-2} (x_{k+2} - x_k) U((\cdot - x_{k+1})_+; x) [x_k, x_{k+1}, x_{k+2}; f],
\]

for all \( f \in \mathcal{F}[a, b] \), \( x \in [a, b] \).

Proof. For fixed \( x \in [a, b] \), we consider the functional \( U(\cdot; x) \in \mathcal{A}_n \). The polynomials

\[
\frac{\ell(t)}{(t-x_0)}, \frac{\ell(t)}{(t-x_0)(t-x_1)}, \frac{\ell(t)}{(t-x_0)(t-x_1)(t-x_2)}, \ldots, \frac{\ell(t)}{(t-x_{n-2})(t-x_{n-1})(t-x_n)},
\]

are linearly independent. Therefore, by virtue of Theorem 3 it follows that there exist \( \alpha(x), \beta(x), a_k(x) \in \mathbb{R}, k = 0, \ldots, n-2 \), such that

\[
U(f; x) = \alpha(x) [x_0; f] + \beta(x) [x_0, x_1; f] + \sum_{k=0}^{n-2} a_k(x) [x_k, x_{k+1}, x_{k+2}; f],
\]

for all \( f \in \mathcal{F}[a, b] \).

Taking successively \( f = e_0 \) and \( f = e_1 \), we obtain

\[
\alpha(x) = U(e_0; x) \quad \text{and} \quad \beta(x) = U(e_1; x) - x_0 U(e_0; x),
\]

hence

\[
U(f; x) = U(e_0, x)f(x_0) + (U(e_1; x) - x_0 U(e_0; x))[x_0, x_1; f] + \sum_{k=0}^{n-2} a_k(x) [x_k, x_{k+1}, x_{k+2}; f], \quad \forall f \in \mathcal{F}[a, b].
\]

In order to compute the numbers \( a_k(x) \) we consider the functions

\[
\varphi_i(t) := (t - x_{i+1})_+,
\]
\(i = 0, \ldots, n - 2\). They satisfy the relations
\[
[x_k, x_{k+1}, x_{k+2}; \varphi_i] = \begin{cases} 
0, & \text{if } k \neq i, \\
\frac{1}{x_{i+2} - x_i}, & \text{if } k = i,
\end{cases}
\]
i, k = 0, \ldots, n - 2. From here, we deduce
\[
U(\varphi_k; x) = \frac{a_k(x)}{x_{k+2} - x_k},
\]
and the proof is completed. \(\square\)

**Corollary 8 (T. Popoviciu [6, p. 151]).** If \(\Delta = \{x_0, \ldots, x_n\}\), then the broken line \(S_\Delta(f)\) associated to \(f\) on \(\Delta\) can be represented in the form
\[
S_\Delta(f)(x) = f(x_0) + (x - x_0) [x_0, x_1; f] + \sum_{k=0}^{n-2} (x_{k+2} - x_k)(x - x_{k+1}) [x_k, x_{k+1}, x_{k+2}; f], \quad x \in \mathbb{R}.
\]

*Proof.* If in Corollary 7 we take \(U := S_\Delta\) and use the fact that \(S_\Delta\) preserves broken lines, i.e.,
\[
\varphi_k(x) = S_\Delta(\varphi_k)(x),
\]
the proof is completed. \(\square\)

**Corollary 9 (Newton Interpolating Formula).** If \(x, x_0, \ldots, x_n\) are distinct points in \([a, b]\), then
\[
f(x) = \sum_{k=0}^n (x - x_0) \cdots (x - x_{k-1}) [x_0, \ldots, x_k; f] + (x - x_0) \cdots (x - x_n) [x_0, \ldots, x_n, x; f],
\]
for all \(f \in F[a, b]\).

*Proof.* Let \(\Delta = \{x_0, \ldots, x_{n+1}\}\) and \(A \in A_n, A(f) := f(x_{n+1})\). The polynomials
\[
\ell(t) = \frac{\ell(t)}{t-x_0} \cdot \frac{\ell(t)}{(t-x_0)(t-x_1)} \cdot \ldots \cdot \frac{\ell(t)}{(t-x_0)(t-x_1) \cdots (t-x_{n+1})},
\]
where \(\ell(t) = (t-x_0)(t-x_1) \cdots (t-x_{n+1})\), are linearly independent. Therefore, by Theorem 3 it follows that there exist \(\lambda_k \in \mathbb{R}, k = 0, \ldots, n + 1\), such that
\[
f(x_{n+1}) = \sum_{k=0}^{n+1} \lambda_k [x_0, \ldots, x_k; f],
\]
for all \(f \in F[a, b]\). In order to calculate the numbers \(\lambda_k\) we consider the functions
\[
\varphi_0(t) := 1, \quad \varphi_i(t) := (t - x_0) \cdots (t - x_{i-1}), \quad i = 1, \ldots, n + 1.
\]
We have
\[
[x_0, \ldots, x_k; \varphi_i] = \delta_{ik}, \quad k, i = 0, \ldots, n + 1.
\]
We obtain
\[ \lambda_i = \varphi_i(x_{n+1}), \quad i = 0, \ldots, n + 1. \]
Consequently,
\[ f(x_{n+1}) = \sum_{k=0}^{n+1} \varphi_k(x_{n+1}) \left[ x_0, \ldots, x_k; f \right], \]
for all \( f \in \mathcal{F}[a, b] \). With \( x_{n+1} = x \) the proof is completed. \( \square \)

REFERENCES


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