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E-CONVEX PROGRAMMING

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. In [6] one shows that some of the results obtained in [5] on E-convex programming are incorrect. In this paper we recover these results in the new hypotheses.

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1. INTRODUCTION

The concepts of E-convex set and E-convex function were introduced in Ref. [5]. For convenience, we remind these definitions.

DEFINITION 1. *A set $M \subseteq \mathbb{R}^n$ is said to be E-convex if there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$(1) \qquad (1-\lambda)E(x) + \lambda E(y) \in M, \text{ for each } x, y \in M \text{ and } 0 \leq \lambda \leq 1.$$

DEFINITION 2. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be E-convex on a set $M \subseteq \mathbb{R}^n$ if there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that M is an E-convex set and*

$$(2) \qquad f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda f(E(x)) + (1-\lambda)f(E(y)),$$

for each $x, y \in M$ and $0 \leq \lambda \leq 1$.

Unfortunately, some of the results obtained in Ref. [5] are incorrect. Indeed, in Ref. [6] one shows that all the results obtained in Ref [5] on E-convex programming are incorrect. In this paper we recover these results in the new hypotheses.

First we remark that, according to Definition 1, each set $M \subseteq \mathbb{R}^n$ is E-convex. Indeed, the empty set is E-convex for any map $E : \mathbb{R}^n \to \mathbb{R}^n$ and, if $M \neq \emptyset$ and $m \in M$, then the map E defined by

$$(3) \qquad E(x) = m, \text{ for all } x \in \mathbb{R}^n,$$

satisfies (1), for all $x, y \in M$ and all $0 \leq \lambda \leq 1$.

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Therefore, we shall restrict this concept to the following definition:

Definition 3. *Let* $E : \mathbb{R}^n \to \mathbb{R}^n$ *be a given map. We say that a subset* M *of* $\mathbb{R}^n$ *is* $E-$*convex if*

$$(1-t)E(x) + tE(y) \in M, \tag{4}$$

for all $x, y \in M$ *and all* $0 \le t \le 1$.

It is also obvious that, according to Definition 2, each function $f : \mathbb{R}^n \to \mathbb{R}^n$ is E-convex on each non-empty subset M of $\mathbb{R}^n$. Indeed, let $m \in M$; then, the map $E : \mathbb{R}^n \to \mathbb{R}^n$ defined by (3) satisfies (2), for all $x, y \in M$ and all $\lambda \in [0, 1]$ (according to the remark above, the set M is E-convex).

Therefore, we shall also restrict this concept to the following definition:

Definition 4. *Let* $M \subseteq \mathbb{R}^n$ *be a non-empty subset of* $\mathbb{R}^n$ *and* $E : \mathbb{R}^n \to \mathbb{R}^n$ *be a map. A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is said to be* E*-convex on* M *if* M *is* E*-convex and*

$$f((1-t)E(x) + tE(y)) \le (1-t)f(E(x)) + tf(E(y)), \tag{5}$$

for all $x, y \in M$ *and all* $0 \le t \le 1$.

Because, in the following, we use the notion of slack 2-convexity with respect to a given set (see [3], or [1]) we remind this notion.

Definition 5. *Let* A *and* B *be two non-empty subsets of* $\mathbb{R}^n$. *We say that* A *is slack 2-convex with respect to* B *if for each* $x, y \in A \bigcap B$ *and each* $t \in [0, 1]$ *such that*

$$(1-t)x + ty \in B,$$

we have

$$(1-t)x + ty \in A.$$

Remark 1. If the set $A \subseteq \mathbb{R}^n$ is slack 2-convex with respect to the set $B \subseteq \mathbb{R}^n$, then the set $A \bigcap C$ is also slack 2-convex with respect to B, for all sets $C \subseteq \mathbb{R}^n$.

2. E-CONVEX PROGRAMMING

Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a function and $M \subseteq \mathbb{R}^n$ be an E-convex set.

We consider the following E-convex programming problem:

$$\text{(P)} \quad \begin{cases} f(x) \to \min \\ g_i(x) \le 0, \ \ i \in \{1, ..., m\} \\ x \in M, \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, ..., m\}$, are E-convex functions.

Theorem 6. *The set of the feasible solutions of Problem* (P), *i.e. the set*

$$S = \{x \in M \,|\, g_i(x) \le 0, \ \ \textit{for all} \ \ i \in \{1, ..., m\}\},$$

and the set

$$S_E = S \bigcap E(M)$$

are slack 2-convex with respect to $E(M)$.

Proof. If we can prove that S is slack 2-convex with respect to $E(M)$ then, in view of Remark 1, we get that S_E is also slack 2-convex with respect to $E(M)$.

Let x', x'' be two elements of $S \bigcap E(M)$ and $t \in [0, 1]$ such that

$$(1-t)x' + tx'' \in E(M).$$

Then we have

$$g_i(x') \leq 0 \text{ and } g_i(x'') \leq 0, \text{ for all } i \in \{1, ..., m\},$$

and there are y', $y'' \in M$ such that

$$x' = E(y') \text{ and } x'' = E(y'').$$

Because of the E-convexity of the set M, we get that

$$(1-t)x' + tx'' = (1-t)E(y') + tE(y'') \in M. \tag{6}$$

Also, for each $i \in \{1, ..., m\}$, we obtain

$$\begin{aligned} g_i((1-t)x' + tx'') &= g_i((1-t)E(y') + tE(y'')) \\ &\leq (1-t)g_i(E(y')) + tg_i(E(y'')) \\ &= (1-t)g_i(x') + tg_i(x'') \\ &\leq 0. \end{aligned} \tag{7}$$

From (6) and (7) we get $(1-t)x' + tx'' \in S$. Therefore, in view of Definition 5, the set S is slack 2-convex with respect to $E(M)$. □

THEOREM 7. *If* $S \subseteq E(M)$, *then the set* S_0 *of the optimal solutions of Problem* (P) *is slack 2-convex with respect to* $E(M)$.

Proof. If S_0 is the empty set, then it is slack 2-convex with respect to $E(M)$. Suppose that it is non-empty and let x', $x'' \in S_0 \bigcap E(M)$. Let $t \in [0, 1]$ such that

$$x = (1-t)x' + tx'' \in E(M).$$

Applying Theorem 6, we get that $x \in S$. Let $f_0 = f(x') = f(x'')$. Because x', $x'' \in E(M)$, there exist y', $y'' \in M$ such that

$$x' = E(y') \text{ and } x'' = E(y'').$$

Because of the E-convexity of the function f, we obtain

$$\begin{aligned} f((1-t)x'+tx'') =& f((1-t)E(y')+tE(y'')) \\ \leq& (1-t)f(E(y')) + tf(E(y'')) \\ =& (1-t)f(x')+tf(x'') \\ =& (1-t)f_0+tf_0 \\ =& f_0. \end{aligned}$$

Therefore $x \in S_0$. □

THEOREM 8. *If* $S \subseteq E(M)$ *and* $x^0 \in intS$ *is a local minimum point of* f *with respect to* S, *then* x^0 *is an optimal solution of Problem* (P).

Proof. As $x^0 \in intS \subseteq E(M)$ is a local minimum point of f, it results that there is a real number $r > 0$, such that

$$B(x^0, r) \subseteq E(M) \tag{8}$$

and

$$f(x) \geq f(x^0), \text{ for all } x \in B(x^0, r) \bigcap S. \tag{9}$$

If we suppose that x^0 is not an optimal solution of Problem (P), then there is $x^* \in S$ such that

$$f(x^*) < f(x^0).$$

Obviously,

$$||x^0 - x^*|| \geq r.$$

Let

$$t = \frac{r}{2||x^0 - x^*||}.$$

Then

$$x = (1-t)x^0 + tx^* \in B(x^0, r), \tag{10}$$

and (8) implies

$$x \in E(M).$$

Applying Theorem 6, we obtain

$$x \in S. \tag{11}$$

On the other hand, the hypothesis $S \subseteq E(M)$ implies that there are $y^0 \in M$ and $y^* \in M$ such that

$$x^0 = E(y^0) \text{ and } x^* = E(y^*).$$

The function f being E-convex, we have

$$\begin{aligned} f(x) &= f((1-t)E(y^0) + tE(y^*)) \\ &\leq (1-t)f(E(y^0)) + tf(E(y^*)) \\ &= (1-t)f(x^0) + tf(x^*) \\ &< f(x^0). \end{aligned}$$

The last strict inequality and the relations (10) and (11) contradict (9). Therefore, x^0 is an optimal solution of Problem (P). □

REFERENCES

[1] Cristescu, G. and Lupşa, L. *Non-connected Convexities and Applications,* Kluwer Academic Publishers, Dordrecht-Boston-London, 2002.

[2] Duca D., Duca E., Lupşa L. and Blaga, R., *E-Convex Functions*, Bulletins for Applied & Computer Mathematics, Budapest 2000; BAM - 1804/2000 XCII, pp. 95–101.

[3] Lupşa, L., *On a convexity notion*, in Seminarul itinerant de ecuaţii funcţionale, aproximare şi convexitate, 1980, 7-8 noiembrie, Timişoara, Timişoara University Publishing House, pp. 127–135, 1980 (in Romanian).

[4] Lupşa, L., *Slack convexity with respect to a given set*, in Seminarul itinerant de ecuaţii funcţionale, aproximare şi convexitate, 1985, Cluj-Napoca, "Babeş-Bolyai" University Publishing House, pp. 107–114, 1985 (in Romanian).

[5] Younnes, E. A., *E-Convex Sets, E-Convex Functions, and E-Convex Programming,* J. Optim. Theory Appl., **109**, no. 2, pp. 439–450, 2001.

[6] Yang, X. M., *On E-Convex Sets, E-Convex Functions, and E-Convex Programming,* J. Optim. Theory Appl., **109**, no. 3, pp. 699–704, 2001.

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