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MINISUM LOCATION PROBLEMS IN DIRECTED NETWORKS

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. We study some location problems in directed networks: we define circular medians and p-circular medians, $p > 1$. We present an algorithm for establishing circular medians. We adopt the definition of network as metric space in the sense of Dearing and Francis (1974).

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1. PRELIMINARY NOTIONS AND RESULTS

Firstly, we recall the definitions of undirected networks as metric space introduced in [1] by P. M. Dearing and R. L. Francis, and also used in [2], [10], [11], [7], [14], [13].

We consider an undirected, connected graph $G = (W, A)$, without loops or multiple edges. To each vertex $w_i \in W = \{w_1, w_2, ..., w_n\}$ we associate a point v_i from $\mathbb{R}^q$, $q \geq 2$. This yields a finite subset $V = \{v_1, v_2, ..., v_n\}$ of $\mathbb{R}^q$, called the vertex set of the network. We also associate to each edge $(w_i, w_j) \in A$ a rectifiable arc $[v_i, v_j] \subset \mathbb{R}^q$ called edge of the network. We assume that any two edges have no interior common points. Consider that $[v_i, v_j]$ has the positive length l_{ij} and denote by U the set of all edges. We define the network $N = (V, U)$ by

$$N = \{x \in \mathbb{R}^q \mid \exists\, (w_i, w_j) \in A \text{ such that } x \in [v_i, v_j]\}.$$

Suppose that for each edge $[v_i, v_j] \in U$ there is a continuous one-to-one mapping $\theta_{ij} : [v_i, v_j] \to [0, 1]$ with $\theta_{ij}(v_i) = 0, \theta_{ij}(v_j) = 1$, and $\theta_{ij}([v_i, v_j]) = [0, 1]$. We denote by T_{ij} the inverse function of θ_{ij}.

We consider an edge $u = [v_i, v_j] \in U$ and the points $x, y \in [v_i, v_j]$. The set of points from the edge $[v_i, v_j]$, between x and y, included x and y, is called closed subedge and is denoted by $[x, y]$. If one or both of x, y are missing we say than the subedge is open in x, or in y or is open and we denote this by

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$(x, y]$, $[x, y)$ or (x, y), respectively. Using θ_{ij}, it is possible to compute the length of $[x, y]$ as

$$l([x, y]) = |\theta_{ij}(x) - \theta_{ij}(y)| \cdot l_{ij}.$$

Particularly we have

$$l([v_i, v_j]) = l_{ij}, l([v_i, x]) = \theta_{ij}(x)\, l_{ij}$$

and

$$l([x, v_j]) = (1 - \theta_{ij}(x))\, l_{ij}.$$

We consider the points $x, y \in N$ and a sequence of edges and at most two subedges

$$[x, v_1], [v_1, v_2], \ldots, [v_{k-1}, v_k], [v_k, y], k \in \mathbb{N}, k \leq n, \quad (1)$$
$$\{v_1, v_2, \ldots, v_k\} \subseteq V$$

in which the vertices are not necessary distinct in twos.

A path $D(x, y)$ between the points $x, y \in N$ is the union of the edges and subedges from the sequence (1). If $x = y$ then the path is called cycle. The length of a path (cycle) is the sum of the lengths of all its component edges and subedges and will be denoted by $l(D(x, y))$. If a path (cycle) contains only distinct vertices then we call it elementary.

If $u = [v_i, v_j]$ is an edge of the network, we say that vertices v_i, v_j are adjacent and u and v_i respectively u and v_j are incident. If two distinct edges are incident with the same vertex then we say that the edges are adjacent.

A network is connected if for any points $x, y \in N$ there is a path $D(x, y) \subset N$.

A vertex v of a network N is called articulation point if $N \backslash \{v\}$ it is not connected.

A connected network without cycles is called tree. In a tree there is a single path $D(x, y)$ for every two points x, y of the tree.

Any subset of a network is called subnetwork. If N' is a subnetwork of a connected network N and v is a vertex from N' then the degree of v in N', $g_{N'}(v)$, is the number of the edges and subedges containing v.

If $V' \subset V$ the induced subnetwork from V' is $N(V') = (V', U')$ where $U' \subset U$ contain all the edges $[v, v'] \in U$ with $v, v' \in V'$.

The blocs of a network N are the maximal induced subnetworks without articulation points.

A network N is called cactus if any two cycles has at most a common vertex.

It is immediately that trees and cycles are particularly cases of cactus.

Let $D^*(x, y)$ be a shortest path between the points $x, y \in N$. This path is also called geodesic.

Definition 1. [1]. *For any $x, y \in N$, the distance from x to y, $d(x, y)$ in the network N is the length of a shortest path from x to y:*

$$d(x, y) = l(D^*(x, y)).$$

Proposition 2. [9]. *The function*

$$d(\cdot,\cdot) : N \times N \to \mathbb{R}, \tag{2}$$
$$d\,(x,y) = l\,(D^*\,(x,y))\,, \forall x,y \in N,$$

is a metric on N.

If in an undirected network $N = (V,U)$, we attach to each edge a sense, we obtain a directed network, and the edges together with the respective senses are called arcs.

If $[v_i, v_j]$ is an arc of directed network then we suppose the sense of the arc is from the vertex v_i to the vertex v_j. If $x, y \in [v_i, v_j]$ and $[x, y]$ has the same sense as the arc $[v_i, v_j]$ then $[x, y]$ is called subarc.

We consider a directed networks $N = (V,U)$, and $x, y \in N$.

A directed path $D(x,y)$ from the point $x \in N$ to the point $y \in N$, is a sequence of arcs and at most two subarcs at the extremities, having the same sense:

$$[x, v_1]\,, [v_1, v_2]\,, \ldots, [v_{k-1}, v_k]\,, [v_k, y]\,, k \in \mathbb{N},$$
$$k \leq n, \{v_1, \ldots, v_k\} \subseteq V, |\{v_1, \ldots, v_k\}| = k.$$

If $x = y$ then the directed path is called circuit.

We denote by $D^*\,(x,y)$ a shortest directed path from x to y.

Definition 3. *A directed network N is called strong connected if there is a directed path from x to y, for every $x, y \in N$.*

A directed network N is called directed cactus if it is obtained from a cactus, attaching to each edge a sense.

The following theorems are immediately implied.

Theorem 4. *A directed cactus is strong connected if and only if every bloc is a circuit.*

Theorem 5. *There is a single directed path from x to y in N if and only if N is a strong connected directed cactus.*

Remark 1. In this sense, in the case of directed networks, strong connected directed cactus are the analogs of trees. $\square$

In the following lines we will endow a strong connected directed network N with a metric space structure.

We consider a strong connected directed network N, the points $x, y \in N$, and $D^*\,(x,y)$ a shortest directed path from x to y.

The function

$$\begin{aligned} d_1(\cdot,\cdot) &: N \times N \to \mathbb{R}, \text{ where} \\ d_1\,(x,y) &= l\,(D^*\,(x,y)) \text{ for every } x,y \in N, \end{aligned} \tag{3}$$

it is not a metric on N since it is not symmetric.

We will define a metric on N, analogously to that introduced by Bohdan Zelinka in [20], for directed graphs.

We define the function

$$d^0(\cdot,\cdot) : N \times N \to \mathbb{R}, \text{ where}$$
$$d^0(x,y) = d_1(x,y) + d_1(y,x), \text{ for every } x, y \in N. \tag{4}$$

Proposition 6. [15]. *The function $d^0(\cdot,\cdot) : N \times N \to \mathbb{R}$ is a metric on N.*

We call the metric d^0, circular metric.

That is $d^0(x,y)$ is equal with the length of a shortest directed path going from x to y and then back to x.

Note that in the mentioned path, called circular path, vertices and arcs may repeat.

We obtain a modality to compute the circular distance when the network is a strong connected directed cactus.

We consider a strong connected directed cactus N.

Theorem 7. [15]. *If x and y are distinct vertices of N then $d^0(x,y)$ is equal with summa of the lengths of all blocs from N which contain arcs of the path $D(x,y)$.*

In the following lines we consider a directed network $N = (V,U)$, a vertex $v_k \in V$, an edge $[v_i, v_j] \in U$ and a point $x = T_{ij}(\theta) \in [v_i, v_j]$, $\theta \in [0,1]$.

Theorem 8. *The function*

$$f_{ij}^k : [0,1] \to \mathbb{R},$$
$$f_{ij}^k(\theta) = d^0(v_k, T_{ij}(\theta)), \quad \theta \in [0,1] \tag{5}$$

is constant on $(0,1)$.

Proof. Indeed, from the definition of circular distance, for every $\theta \in (0,1)$ we obtain:

$$\begin{aligned} f_{ij}^k(\theta) &= d^0(v_k, T_{ij}(\theta)) \\ &= d_1(v_k, v_i) + \theta l_{ij} + (1-\theta) l_{ij} + d_1(v_j, v_k) \\ &= d_1(v_k, v_i) + l_{ij} + d_1(v_j, v_k) \end{aligned}$$

hence the function f_{ij}^k is constant on $(0,1)$. □

We denote by c_{ij}^k the constant value of the function f_{ij}^k on $(0,1)$.

Theorem 9. $f_{ij}^k(0) \le c_{ij}^k$ *and* $f_{ij}^k(1) \le c_{ij}^k$.

Proof. Indeed,

$$\begin{aligned} f_{ij}^k(0) =&d^0(v_k, T_{ij}(0)) \\ =&d^0(v_k, v_i) \\ =&d_1(v_k, v_i) + d_1(v_i, v_k) \\ \le&d_1\,(v_k, v_i) + l_{ij} + d_1\,(v_j, v_k) \\ =&c_{ij}^k, \end{aligned}$$

and

$$\begin{aligned} f_{ij}^k(1) =&d^0(v_k, T_{ij}(1)) \\ =&d^0(v_k, v_j) \\ =&d_1(v_k, v_j) + d_1(v_j, v_k) \\ \le&d_1\,(v_k, v_i) + l_{ij} + d_1\,(v_j, v_k) \\ =&c_{ij}^k. \end{aligned}$$

□

2. A MINISUM LOCATION PROBLEM IN DIRECTED NETWORK

Locations on networks were initiated by the work of Hakimi [4] who formulated the minimax and minisum problem in their most current form, presented a geometrical algorithm for the minimax problem and proved his vertex optimality theorem, namely the set of vertices always contains an optimal solution to the minisum problem. The literature on network location problem has grown rapidly since.

The network location problem may be formulated as follows: find the location of a number of points on a given network (called facilities) so as to provide goods or services to a specified set of potential users and to optimize one or several criteria. Examples of facilities are plants, warehouses, deposits, schools, hospitals, bus stops, mail boxes, switching centers in communication networks, etc.

The problem is motivated by a number of potential applications. For example: several plants are to be set up at some points of a transportation system to minimize production and shipment costs, a computer is established at some point of a communication network to minimize transmission costs from and towards peripheral units. When we choose the location of an emergency service we are concerned with the largest distance from this location to the site occupied by a potential user.

The minimax location problem is to determine the location of a single facility for which the largest distance from this location to the site occupied by a potential user is minimized. Restriction on possible locations of the facility and users to vertices or to the entire network generates four types of problems. In the first case, a vertex must be found for which the distance to the farthest vertex is minimized. The solution vertex is called center. This problem was studied by Hakimi in [4]. The center problem is very easily solved by comparison with distance vectors once the distance matrix between vertices is known. In the second case a vertex must be determined so as distance to the most remote point of the network is minimized (Minieka [17]). The solution vertex is called general center. In the third case a point of the network must be found, minimizing the maximum distance to any vertex (Hakimi [4] and Kariv and Hakimi [8]). Finding such a solution point, called absolute center, is more difficult. In [4] is presented a method to find the absolute center set. In the fourth and most general case, both the locations of the facility and of the users can be chosen any where on the network. Frank [3] presents the principle of an algorithm to determine a solution point, called continuous center. He proposed to extend Hakimi's approach for the absolute center and a similar proposal was made by Minieka [17]. Concerning the continuous center set, Labbé [9] followed Frank's work by providing a detailed discussion of the problem including the treatment of an overlooked case, a polynomial implementation of the resulting algorithm and a series of rules designed to accelerate the algorithm.

In [12] Labbé and Louveaux presented a selective bibliography regarding location problems.

In [16] we studied some location problems in directed networks: we defined circular centers, circular absolute centers and circular continuous centers. We presented algorithms for establishing circular absolute centers and circular continuous centers.

In the following lines we will study a minisum location problem.

We consider a set of users located at the vertices of a strong connected directed network $N = (V, U)$, $|V| = n$.

The minisum location problem (or median problem) in directed network N, considered in the following lines, is to determine a point on N such that the summa of circular distance from this point to the users is minimized.

We consider the function

$$F : N \to \mathbb{R}, F(x) = \sum_{i=1}^{n} d^0(v_i, x), \forall x \in N.$$

Definition 10. *A point $m \in N$ is a circular median if for any $x \in N$ the following inequality is satisfied*

$$F(m) \leq F(x).$$

The set of all circular median is denoted by A^0.

Theorem 11. *The set of vertices contain a circular median.*

Proof. We consider an edge $[v_i, v_j] \in U$ and $v_k \in V$. We recall that the function (5), $f_{ij}^k : [0,1] \to \mathbb{R}$ is constant on $(0,1)$, the values of this function in 0 and 1 being lesser or equal with its constant value on $(0,1)$, $\forall v_k \in V$

(Theorems 8, 9). Consequently the restriction of the function $F : N \to \mathbb{R}$, to $[v_i, v_j]$ is constant on $]v_i, v_j[$ and $F(x) \geq \min\{F(v_i), F(v_j)\}$, $\forall x \in\,]v_i, v_j[$. Hence the function F attains its minimum in V. $\square$

The following two theorems are immediately implied.

THEOREM 12. *If all the functions* $f_{ij}^k : [0,1] \to \mathbb{R}$ *are not continuous in* 0 *and* 1, $\forall\, [v_i, v_j] \in U$ *and* $\forall v_k \in V$, *then any circular median is a vertex.*

THEOREM 13. *If there is a point* $x \in [v_i, v_j] \in U$ *which is a circular median then all the points of the arc* $[v_i, v_j]$ *are circular medians.*

To determine circular medians we can use the following algorithm.

ALGORITHM 14. *1. Compute* $F(v_i), \forall v_i \in V$.
2. Determine $F(m) = \min\{F(v_i) \mid v_i \in V\}$ *and denote by* C_1 *the set of all vertices for which the minimum is attains.*
3. For each vertex $v \in C_1$ *consider on every incident arc a point* x *different by extremities and compute* $F(x)$. *If* $F(x) = F(m)$ *then all the points of respective arc are circular medians. We denote by* C_v *the set of all these determined points, if exist. Else* $C_v = \emptyset$.
4. $C_2 = \bigcup\limits_{v \in V} C_v$.
5. $A^0 = C_1 \cup C_2$.

Finally, the vertex optimality property can be extended to the location problems of p facilities, $p > 1$. Assume that the users go to the closest among these p facilities. Then an optimal solution, called a p-circular median is a set of p points $X_p = \{x_1, x_2, \ldots, x_p\}$ such that for any set $Y_p = \{y_1, y_2, \ldots, y_p\}$, $|Y_p| = p$,

$$F(X_p) = \sum_{i=1}^{n} \min\left\{d^0(v_i, x), x \in X_p\right\} \leq F(Y_p) = \sum_{i=1}^{n} \min\left\{d^0(v_i, y), y \in Y_p\right\}.$$

In this case, there always exists a subset of vertices which is a p-circular median.

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