REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 34 (2005) no. 1, pp. 3–15
ictp.acad.ro/jnaat

A SEMILOCAL CONVERGENCE ANALYSIS FOR THE METHOD OF TANGENT PARABOLAS

IOANNIS K. ARGYROS*

Abstract. We present a semilocal convergence analysis for the method of tangent parabolas (Euler–Chebyshev) using a combination of Lipschitz and center Lipschitz conditions on the Fréchet derivatives involved. This way we produce a majorizing sequence which converges under weaker conditions than before. The error bounds obtained are more precise and the information of the location of the solution better than in earlier results.

MSC 2000. 65B05, 65G99, 65J15, 65N30, 65N35, 47H17, 49M15.
Keywords. Banach space, tangent parabola, Euler–Chebyshev method, majorizing sequence, Fréchet-derivative, Lipschitz-center conditions.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where F is a twice-Fréchet differentiable operator on an open convex subset D of a Banach space X with values in a Banach space Y.

The method of tangent parabolas (Euler–Chebyshev)

(2)
$$x_{n+1} = x_n - \left\{ I + \tfrac{1}{2}\Gamma_n F''(x_n)\Gamma_n F(x_n) \right\} \Gamma_n F(x_n),\ \Gamma_n = F'(x_n)^{-1} \quad (n \geq 0)$$

is one of the best known cubically convergent iterative procedures for solving nonlinear equations like (1). Here $F'(x_n) \in L(X,Y)$, $F''(x_n) \in L(X, L(X,Y))$ denote the first and second Fréchet derivatives of operator F evaluated at $x = x_n$ $(n \geq 0)$ [3], [7].

Semilocal convergence results under Lipschitz conditions on the second Fréchet-derivative have been given by Necepurenko [9], Mertvecova [8], Safiev [10], Schwetlick [11], Kanno [6], Yamamoto [12], Argyros [1]–[3], Gutiérrez et al. [4], [5]. Discretized versions of this method have been considered in [2], [3].

Here we provide a semilocal convergence analysis based on Lipschitz and center-Lipschitz conditions on the first and second Fréchet-derivatives of F.

*Cameron University, Department of Mathematical Sciences, Lawton, OK 73505, USA, e-mail: ioannisa@cameron.edu.

This way existing convergence conditions are finer and the information on the location of the solution more precise than before.

2. CONVERGENCE ANALYSIS

We need the following results on majorizing sequences.

THEOREM 1. *Let η, ℓ_i, $i = 0, 1, \ldots, 4$ be non-negative parameters. Define scalar sequence $\{t_n\}$ $(n \geq 0)$ by*

$$\begin{aligned} t_0 &= 0, \quad t_1 = \left(1 + \tfrac{1}{2}\ell_0\eta\right)\eta = \eta_0, \\ t_{n+2} - t_{n+1} &= \left[1 + \tfrac{1}{2}\tfrac{\ell_0+\ell_3 t_{n+1}}{(1-\ell_1 t_{n+1})^2}\eta_{n+1}\right]\tfrac{\eta_{n+1}}{1-\ell_1 t_{n+1}}, \end{aligned} \tag{3}$$

where

$$\eta_{n+1} = \tfrac{1}{2}\left\{\tfrac{\ell_2}{4}\tfrac{(\ell_0+\ell_3 t_\eta)^2}{(1-\ell_1 t_n)^6}\eta_\eta + \ell_2\tfrac{\ell_0+\ell_3 t_n}{(1-\ell_1 t_n)^4} + \tfrac{\ell_4}{3}\tfrac{1}{(1-\ell_1 t_n)^3}\right\}\eta_\eta^3,$$

and parameter α by

$$\alpha = \left[\tfrac{\ell_2}{4}\tfrac{(\ell_0+2\ell_3\eta_0)^2}{(1-2\ell_1\eta_0)^6}\eta_0 + \ell_2\tfrac{\ell_0+2\ell_3\eta_0}{(1-2\ell_1\eta_0)^4} + \tfrac{\ell_4}{3(1-2\ell_1\eta_0)^3}\right]\eta_0^2. \tag{4}$$

Assume:

$$\begin{aligned} \ell_0\eta &\leq 2, \\ 2\ell_1\eta_0 &< 1, \end{aligned}$$

and

$$\alpha \leq \min\{1, \alpha_0\},$$

where α_0 is the positive solution of quadratic equation

$$\tfrac{1}{4}\tfrac{\ell_0+2\ell_3\eta_0}{(1-2\ell_1\eta_0)^3}\eta_0 t^2 + \tfrac{t}{1-2\ell_1\eta_0} - 1 = 0. \tag{5}$$

Then, sequence $\{t_n\}$ $(n \geq 0)$ is non-decreasing, bounded above by

$$t^{**} = 2\eta_0,$$

and converges to t^* such that

$$0 \leq t^* \leq t^{**}. \tag{6}$$

Moreover, the following error bounds hold for all $n \geq 0$:

$$0 \leq t_{n+2} - t_{n+1} \leq \tfrac{1}{2}(t_{n+1} - t_n) \leq \left(\tfrac{1}{2}\right)^{n+1}\eta_0.$$

Proof. Using induction on k we show:

$$\eta_{k+1} \leq \tfrac{1}{2}\eta_k, \tag{7}$$

$$t_{k+1} - t_k \geq 0, \tag{8}$$

and

$$1 - \ell_1 t_{k+1} > 0. \tag{9}$$

For $k = 0$ (7)–(9) hold by the initial conditions. By (3) we then get

$$t_2 - t_1 \leq \tfrac{\alpha}{2}(t_1 - t_0) \leq \tfrac{1}{2}(t_1 - t_0).$$

Let us assume (7)–(9) hold for all $k \leq n+1$. We can easily obtain from (3) that

$$t_{k+1} \leq \frac{1-\left(\frac{1}{2}\right)^{k+1}}{1-\frac{1}{2}}\eta_0 = 2\left[1-\left(\frac{1}{2}\right)^{k+1}\right]\eta_0 \leq t^{**}.$$

Moreover we have,

$$t_{k+2}-t_{k+1} \leq \alpha^{k+1}\left[1+\frac{1}{4}\frac{\ell_0+2\ell_3\eta_0}{(1-2\ell_1\eta_0)^2}\eta_0\alpha\right]\frac{1}{1-2\ell_1\eta_0}\left(\frac{1}{2}\right)^{k+1}\eta_0 \leq \left(\frac{1}{2}\right)^{k+1}\eta_0$$

by the choice of α and α_0 (see (4) and (5)).

Furthermore we have

$$\begin{aligned} t_{k+2} &\leq 2\left[1-\left(\frac{1}{2}\right)^{k+2}\right]\eta_0 \leq t^{**}, \\ \ell_1 t_{k+2} &\leq 2\ell_1\eta_0 < 1, \end{aligned}$$

and

$$t_{k+2}-t_{k+1} \geq 0.$$

The induction for (7)–(9) is now complete. Hence, sequence $\{t_n\}$ $(n \geq 0)$ is bounded above by t^{**}, non-decreasing and as such it converges to some t^* satisfying (6). That completes the proof of Theorem 1. □

Similarly we show the next two theorems:

THEOREM 2. *Let η, ℓ_0, ℓ_3, ℓ_4 be non-negative parameters. Define scalar sequence $\{s_n\}$ $(n \geq 0)$ by*

$$\begin{aligned} s_0 &= 0, \quad s_1 = \left(1+\tfrac{1}{2}\ell_0\eta\right)\eta = \eta_0, \\ s_{n+2}-s_{n+1} &= \left[1+\frac{1}{2}\frac{(\ell_0+\ell_3 s_{n+1})\eta_{n+1}}{(1-\ell_0 s_{n+1}-\ell_3 s_{n+1}^2)^2}\right]\frac{\eta_{n+1}}{1-\ell_0 s_{n+1}-\ell_3 s_{n+1}^2}, \end{aligned}$$

where

$$\eta_{n+1} = \frac{1}{2}\left\{\frac{1}{2}\frac{(\ell_0+\ell_3 s_n)^2}{(1-\ell_0 s_n-\frac{\ell_3}{2}s_n^2)^4}+\frac{\ell_3}{6}\frac{(\ell_0+\ell_3 s_n)\eta_\eta}{(1-\ell_0 s_n-\ell_3 s_n^2)^5}+\frac{\ell_4}{3}\frac{1}{(1-\ell_0 s_n-\ell_3 s_n^2)^3}\right\}\eta_\eta^3$$

and parameter α by

$$\alpha = \left\{\frac{1}{2}\frac{(\ell_0+2\ell_3\eta_0)^2}{(1-2\ell_0\eta_0-2\ell_3\eta_0^2)^4}+\frac{\ell_3}{6}\frac{(\ell_0+2\ell_3\eta_0)\eta_0}{(1-2\ell_0\eta_0-2\ell_3\eta_0^2)^5}+\frac{\ell_4}{3(1-2\ell_0\eta_0-2\ell_3\eta_0^2)^3}\right\}\eta_0^2.$$

Assume:

(10) $$2(\ell_0+\ell_3\eta_0)\eta_0 < 1,$$

and

(11) $$\alpha \leq \min\{1, \alpha_0\},$$

where α_0 is the positive solution of quadratic equation

$$\frac{1}{4}\frac{\ell_0+2\ell_3\eta_0}{(1-2\ell_0\eta_0-2\ell_3\eta_0^2)^3}\eta_0 t^2+\frac{t}{1-2\ell_0\eta_0-2\ell_3\eta_0^2}-1=0.$$

Then, sequence $\{s_n\}$ $(n \geq 0)$ is non-decreasing, bounded above by

$$s^{**} = 2\eta_0,$$

and converges to s^* *such that*

$$0 \leq s^* \leq s^{**}.$$

Moreover, the following error bounds hold for all $n \geq 0$

$$0 \leq s_{n+2} - s_{n+1} \leq \tfrac{1}{2}(s_{n+1} - s_n) \leq \left(\tfrac{1}{2}\right)^{n+1} \eta_0.$$

THEOREM 3. *Let* η, ℓ_0, ℓ_1, ℓ_3, ℓ_4 *be non-negative parameters. Define scalar sequence* $\{v_n\}$ $(n \geq 0)$ *by*

$$\begin{aligned} v_0 &= 0, \quad v_1 = \left(1 + \tfrac{1}{2}\ell_0\eta\right)\eta = \eta_0, \\ v_{n+2} - v_{n+1} &= \left[1 + \tfrac{1}{2}\tfrac{(\ell_0+\ell_3 v_{n+1})\eta_{n+1}}{(1-\ell_1 v_{n+1})^2}\right] \tfrac{\eta_{n+1}}{1-\ell_1 v_{n+1}}, \end{aligned}$$

where

$$\eta_{n+1} = \tfrac{1}{2}\left\{\tfrac{1}{2}\tfrac{(\ell_0+\ell_3 v_n)^2}{(1-\ell_1 v_n)^4} + \tfrac{\ell_3}{6}\tfrac{(\ell_0+\ell_3 v_n)\eta_n}{(1-\ell_1 v_n)^5} + \tfrac{\ell_4}{3(1-\ell_1 v_n)^3}\right\}\eta_\eta^3,$$

and parameter α *by*

$$\alpha = \left\{\tfrac{1}{2}\tfrac{(\ell_0+2\ell_3\eta_0)^2}{(1-2\ell_1\eta_0)^4} + \tfrac{\ell_3}{6}\tfrac{(\ell_0+2\ell_3\eta_0)\eta_0}{(1-2\ell_1\eta_0)^5} + \tfrac{\ell_4}{3(1-2\ell_1\eta_0)^3}\right\}\eta_0^2.$$

Assume:

$$\begin{aligned} \ell_0\eta &\leq 2, \\ 2\ell_1\eta_0 &< 1, \end{aligned}$$

and

$$\alpha \leq \min\{1, \alpha_0\},$$

where α_0 *is the positive solution of quadratic equation*

$$\tfrac{1}{4}\tfrac{\ell_0+2\ell_3\eta_0}{(1-2\ell_1\eta_0)^3}\eta_0 t^2 + \tfrac{t}{1-2\ell_1\eta_0} - 1 = 0.$$

Then, sequence $\{v_n\}$ $(n \geq 0)$ *is non-decreasing, bounded above by*

$$v^{**} = 2\eta_0,$$

and converges to v^* *such that*

$$0 \leq v^* \leq v^{**}.$$

Moreover, the following error bounds hold for all $n \geq 0$

$$0 \leq v_{n+2} - v_{n+1} \leq \tfrac{1}{2}(v_{n+1} - v_n) \leq \left(\tfrac{1}{2}\right)^{n+1} \eta_0. \tag{12}$$

We can show the main semilocal convergence theorem for method (2).

THEOREM 4. *Let $F\colon D \subseteq X \to Y$ be a twice Fréchet-differentiable operator. Assume: there exist a point $x_0 \in D$ and non-negative parameters η, ℓ_i, $i = 0, 1, \ldots, 4$ such that*

$$F'(x_0)^{-1} \in L(Y, X), \tag{13}$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta, \tag{14}$$

$$\|F'(x_0)^{-1}F''(x_0)\| \leq \ell_0, \tag{15}$$

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq \ell_1\|x - x_0\|, \tag{16}$$

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq \ell_2\|x - y\|, \tag{17}$$

$$\|F'(x_0)^{-1}[F''(x) - F''(x_0)]\| \leq \ell_3\|x - x_0\|, \tag{18}$$

and

$$\|F'(x_0)^{-1}[F''(x) - F''(y)]\| \leq \ell_4\|x - y\| \quad \text{for all } x, y \in D. \tag{19}$$

Moreover, hypotheses of Theorem 1 hold, and

$$\overline{U}(x_0, t^*) = \{x \in X \mid \|x - x_0\| \leq t^*\} \subseteq D.$$

Then the method of tangent parabolas $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $\overline{U}(x_0, t^*)$ for all $n \geq 0$ and converges to a solution $x^* \in \overline{U}(x_0, t^*)$ of equation $F(x) = 0$. Moreover, the following error bounds hold for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n,$$

and

$$\|x_n - x^*\| \leq t^* - t_n.$$

Furthermore, if there exists $R \geq t^*$ such that

$$U(x_0, R) \subseteq D$$

and

$$\ell_1(t^* + R) \leq 2, \tag{20}$$

the solution x^* is unique in $U(x_0, R)$.

Proof. We prove:

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k \tag{21}$$

and

$$\overline{U}(x_{k+1}, t^* - t_{k+1}) \subseteq U(x_k, t^* - t_k) \;\text{ hold for all } k \geq 0. \tag{22}$$

For every $z \in \overline{U}(x_1, t^* - t_1)$

$$\|z - x_0\| \leq \|z - x_1\| - \|x_1 - x_0\| \leq t^* - t_1 + t_1 = t^* - t_0$$

implies $z \in \overline{U}(x_0, t^* - t_0)$. Note also that

$$\begin{aligned} \|x_1 - x_0\| &\leq \left[1 + \tfrac{1}{2}\|F'(x_0)^{-1}F''(x_0)\|\,\|F'(x_0)^{-1}F(x_0)\|\right]\|F'(x_0)^{-1}F(x_0)\| \\ &\leq \left(1 + \tfrac{1}{2}\ell_0\eta\right)\eta = \eta_0. \end{aligned}$$

Since also

$$\|x_1 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \le \eta \le t_1 \quad \text{(by (3))}$$

(21) and (22) hold for $k = 0$. Given they hold for $n = 0, 1, \ldots, k$, then

$$\|x_{k+1} - x_0\| \le \sum_{i=1}^{n+1} \|x_i - x_{i-1}\| \le \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} - t_0 = t_{k+1}$$

and

$$\|x_k + \theta(x_{k+1} - x_k) - x_0\| \le t_k + \theta(t_{k+1} - t_k) < t^*, \quad \theta \in [0,1].$$

It follows from (16)

(23)
$$\|F'(x_0)^{-1}[F'(x_{k+1}) - F'(x_0)]\| \le \ell_1 \|x_{k+1} - x_0\| \le \ell_1 t_{k+1} \le 2\ell_1 \eta_0 < 1 \ \text{(by (7))},$$

and the Banach Lemma on invertible operators [7] that the inverse $F'(x_{k+1})^{-1}$ exists and

$$\|F'(x_{k+1})^{-1}F'(x_0)\| \le [1 - \ell_1\|x_{k+1} - x_0\|]^{-1} \le (1 - \ell_1 t_{k+1})^{-1}.$$

Set

$$y_k = x_k - F'(x_k)^{-1}F(x_k). \tag{24}$$

Then we get from (2)

$$x_{k+1} = y_k - \tfrac{1}{2}F'(x_k)^{-1}F''(x_k)(y_k - x_k)^2. \tag{25}$$

Using (2), (24) and (25) as in [3] we get the approximation

$$\begin{aligned} F(x_{k+1}) = &\int_0^1 [F'(y_k + \theta(x_{k+1} - x_k)) - F'(y_k)](x_{k+1} - y_k)\mathrm{d}t \\ &+ (F'(y_k) - F'(x_k))(x_{k+1} - y_k) \\ &+ \int_0^1 [F''(x_k + \theta(y_k - x_k)) - F''(x_n)](1-t)\mathrm{d}t(y_k - x_k)^2. \end{aligned} \tag{26}$$

By composing both sides of (26) by $F'(x_0)^{-1}$ and setting

$$\overline{\eta}_k = \|F'(x_0)^{-1}F(x_k)\| \quad (k \ge 1), \ \overline{\eta}_0 = \eta_0$$

we get in turn

$$\begin{aligned}
\overline{\eta}_{k+1} \leq\ & \ell_2 \int_0^1 t \Big\| \tfrac{1}{2}[F'(x_k)^{-1}F'(x_0)][F'(x_0)^{-1}(F''(x_k) - F''(x_0) + F''(x_0)] \\
& \qquad \cdot\{[F'(x_k)^{-1}F'(x_0)][F'(x_0)^{-1}F'(x_k)]\}^2 \Big\|^2 \mathrm{d}t \\
& + \ell_2 \|[F'(x_k)^{-1}F'(x_0)][F'(x_0)^{-1}F(x_k)]\| \\
& \quad \cdot \Big\| \tfrac{1}{2}[F'(x_k)^{-1}F'(x_0)]F'(x_0)^{-1}[F''(x_k) - F''(x_0) + F'(x_0)] \\
& \qquad \cdot\{[F'(x_k)^{-1}F'(x_0)][F'(x_0)^{-1}F(x_k)]\}^2 \Big\|
\end{aligned}$$

$$+\ \tfrac{\ell_4}{6}\|[F'(x_k)^{-1}F'(x_0)][F'(x_0)^{-1}F(x_k)]\|^3$$

$$\leq\ \tfrac{\ell_2}{8}\left[\tfrac{(\ell_0+\ell_3\|x_k-x_0\|)^2}{(1-\ell_1\|x_k-x_0\|)^6}\right]\overline{\eta}_k^4+\tfrac{\ell_2}{2}\tfrac{\ell_0+\ell_3\|x_k-x_0\|}{(1-\ell_1\|x_k-x_0\|)^4}\overline{\eta}_k^3+\ \tfrac{\ell_4\overline{\eta}_k^3}{6(1-\ell_1\|x_k-x_0\|)^3}$$

$$\leq\ \tfrac{\ell_2}{8}\left[\tfrac{(\ell_0+\ell_3 t_k)^2}{(1-\ell_1 t_k)^6}\right]\eta_k^4+\tfrac{\ell_2}{2}\tfrac{\ell_0+\ell_3 t_k}{(1-\ell_1 t_k)^4}\eta_k^3+\tfrac{\ell_4\eta_k^3}{6(1-\ell_1 t_k)^3}\,. \tag{27}$$

Hence, we obtain from (2), (14)–(19) and (27)

$$\|x_{k+2}-x_{k+1}\|\leq$$

$$\leq\Big[1+\tfrac{1}{2}\|F'(x_{k+1})^{-1}F'(x_0)\|\cdot\|F'(x_0)^{-1}(F''(x_{k+1})-F''(x_0)+F''(x_0)\|$$

$$\cdot\,\|F'(x_{k+1})^{-1}F'(x_0)\|\cdot\|F'(x_0)^{-1}F(x_{k+1})\|\Big]\|F'(x_{k+1})^{-1}F'(x_0)\|$$

$$\cdot\,\|F'(x_0)^{-1}F(x_{k+1})\|$$

$$\leq\left[1+\tfrac{1}{2}\tfrac{\ell_0+\ell_3\|x_{k+1}-x_0\|}{(1-\ell_1\|x_{k+1}-x_0\|)^2}\overline{\eta}_{k+1}\right]\tfrac{\overline{\eta}_{k+1}}{1-\ell_1\|x_{k+1}-x_0\|}$$

$$\leq\left[1+\tfrac{1}{2}\tfrac{\ell_0+\ell_3 t_{k+1}}{(1-\ell_1(t_{k+1})^2}\eta_{k+1}\right]\tfrac{\eta_{k+1}}{1-\ell_1 t_{k+1}}=t_{k+2}-t_{k+1}, \tag{28}$$

which together with (21) show (16) for all $n\geq 0$.

Thus for every $z\in\overline{U}(x_{k+2},t^*-t_{k+2})$ we have

$$\|z-x_{k+1}\|\leq\|z-x_{k+2}\|+\|x_{k+2}-x_{k+1}\|\leq t^*-t_{k+2}+t_{k+2}-t_{k+1}=t^*-t_{k+1}.$$

That is,

$$z\in\overline{U}(x_{k+1},t^*-t_{k+1}). \tag{29}$$

Estimates (28) and (29) imply that (21) and (22) hold for $n=k+1$. By induction the proof of (21) and (22) is completed.

Theorem 1 implies $\{t_n\}$ $(n\geq 0)$ is a Cauchy sequence. From (21) and (22) $\{x_n\}$ $(n\geq 0)$ becomes a Cauchy sequence too, and as such it converges to some $x^*\in\overline{U}(x_0,t^*)$ (since $\overline{U}(x_0,t^*)$ is a closed set) such that

$$\|x_k-x^*\|\leq t^*-t_k. \tag{30}$$

The combination of (21) and (30) yields $F(x^*)=0$. Finally to show uniqueness let y^* be a solution of equation $F(x)=0$ in $U(x_0,R)$. It follows from (16) the estimate

$$\left\|F'(x_0)^{-1}\int_0^1[F'(y^*+\theta(x^*-y^*))-F'(x_0)]\right\|\mathrm{d}\theta\leq$$

$$\leq\ell_1\int_0^1\|y^*+\theta(x^*-y^*)-x_0\|\mathrm{d}\theta$$

$$\leq\ \ell_1\int_0^1[\theta\|x^*-x_0\|+(1-\theta)\|y^*-x_0\|]\mathrm{d}\theta<\tfrac{\ell_1}{2}(t^*+R)<1,\ \text{(by (20))},$$

and the Banach Lemma on invertible operators that linear operator

$$L=\int_0^1 F'(y^*+\theta(x^*-y^*))\mathrm{d}\theta$$

is invertible.

Using the identity

$$0 = F(x^*) - F(y^*) = L(x^* - y^*)$$

we deduce

$$x^* = y^*.$$

That completes the proof of the theorem. □

THEOREM 5. *Let $F\colon D \subseteq X \to Y$ be a twice Fréchet-differentiable operator. Assume: there exist a point $x_0 \in D$ and non-negative parameters η, ℓ_0, ℓ_3, ℓ_4 such that*

$$\begin{aligned} F'(x_0)^{-1} &\in L(Y,X), \\ \|F'(x_0)^{-1}F(x_0)\| &\leq \eta, \\ \|F'(x_0)^{-1}F''(x_0)\| &\leq \ell_0, \\ \|F'(x_0)^{-1}[F''(x) - F''(x_0)]\| &\leq \ell_3\|x - x_0\| \end{aligned}$$

and

$$\|F'(x_0)^{-1}[F''(x) - F''(y)]\| \leq \ell_4\|x - y\| \quad \text{for all } x, y \in D.$$

Moreover, hypotheses of Theorem 2 hold, and

$$\overline{U}(x_0, s^*) \subseteq D.$$

Then the method of tangent parabolas $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $\overline{U}(x_0, s^*)$ for all $n \geq 0$ and converges to a solution $x^* \in \overline{U}(x_0, s^*)$ of equation $F(x) = 0$.

Moreover, the following error bounds hold for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq s_{n+1} - s_n,$$

and

$$\|x_n - x^*\| \leq s^* - s_n.$$

Furthermore, if there exists $R_1 \geq s^*$ such that

$$U(x_0, R_1) \subseteq D,$$

and

$$\gamma = \tfrac{1}{2}\left[\ell_0 + \tfrac{\ell_3}{4}(R_1 + s^*)\right] \in [0, 1],$$

the solution x^* is unique in $U(x_0, R_1)$.

Proof. It follows along the lines of Theorem 4 but instead of (23) we use

$$\begin{aligned} &\|F'(x_0)^{-1}[F'(x_0) - F'(x_{n+1})]\| \leq \\ &\leq \int_0^1 \|F'(x_0)^{-1}\{F''[x_0 + \theta(x_{k+1} - x_0)] - F''(x_0)\}\mathrm{d}\theta(x_{k+1} - x_0)\| \\ &\quad + \|F'(x_0)^{-1}F''(x_0)(x_{k+1} - x_0)\| \\ &\leq \tfrac{1}{2}\ell_3\|x_{k+1} - x_0\|^2 + \ell_0\|x_{k+1} - x_0\| \leq \tfrac{\ell_3}{2}s_{k+1}^2 + \ell_0 s_{k+1} \\ &\leq 2\ell_3\eta_0^2 + 2\ell_0\eta < 1 \quad \text{by (10)}, \end{aligned} \tag{31}$$

so

$$\begin{aligned}\|F'(x_{k+1})^{-1}F'(x_0)\| &\le \left[1-\ell_0\|x_{k+1}-x_0\|-\tfrac{1}{2}\ell_3\|x_{k+1}-x_0\|^2\right]^{-1}\\ &\le \left(1-\ell_0 s_{k+1}-\tfrac{\ell_3}{2}s_{k+1}^2\right)^{-1}.\end{aligned}$$

Moreover, instead of (26) we use

$$\begin{aligned}F(x_{k+1}) &= \int_0^1 [F'(y_k+t(x_{k+1}-y_k))-F'(x_k)](x_{k+1}-y_k)\\ &\quad+\int_0^1 [F''(x_k+t(y_k-x_k))-F''(x_k)](1-t)\mathrm{d}t(y_k-x_k)^2\\ &= \int_0^1\int_0^1 F''[x_k+\theta t(y_k-x_k)]t(y_k-x_k)\mathrm{d}\theta(x_{k+1}-y_k)\mathrm{d}t\\ &\quad+\int_0^1 [F''(x_k+t(y_k-x_k))-F''(x_k)](1-t)\mathrm{d}t(y_k-x_k)^2,\end{aligned}$$

which as in (27) leads to

$$\begin{aligned}\overline{\eta}_{k+1} &\le \tfrac{1}{4}(\ell_0+\ell_3\|x_k-x_0\|)\|F'(x_k)^{-1}F(x_k)\|^3\|F'(x_k)^{-1}F''(x_k)\|\\ &\quad+\tfrac{1}{12}\ell_3\|F'(x_k)^{-1}F(x_k)\|^3\|F'(x_k)^{-1}F''(x_k)\|+\tfrac{\ell_4}{6}\|F'(x_k)^{-1}F(x_k)\|^3\\ &\le \tfrac{1}{4}\tfrac{(\ell_0+\ell_3\|x_k-x_0\|)^2\overline{\eta}_k^3}{\left[1-\ell_0\|x_k-x_0\|-\frac{\ell_3}{2}\|x_k-x_0\|^2\right]^4}+\tfrac{\ell_3}{12}\tfrac{(\ell_0+\ell_3\|x_k-x_0\|)\overline{\eta}_k^4}{\left(1-\ell_0\|x_k-x_0\|-\frac{\ell_3}{2}\|x_k-x_0\|^2\right)^5}\\ &\quad+\tfrac{\ell_4}{6}\tfrac{\overline{\eta}_k^3}{\left(1-\ell_0\|x_k-x_0\|-\frac{\ell_3}{2}\|x_k-x_0\|^2\right)^3}.\end{aligned}$$

The rest follows as in Theorem 4 until the uniqueness part.

Let y^* be a solution of equation $F(x)=0$ in $U(x_0,R_1)$. For $z\in U(x_0,R_1)$ we have

$$\begin{aligned}&\|F'(x_0)^{-1}[F'(z)-F'(x_0)]\| =\\ &=\left\|F'(x_0)^{-1}\int_0^1 F''(x_0+\theta_1(z-x_0))(z-x_0)\mathrm{d}\theta_1\right\|\\ &\le \left\|F'(x_0)^{-1}\int_0^1 [F''(x_0+\theta_1(z-x_0))-F''(x_0)]\right\|\mathrm{d}\theta_1\|z-x_0\|\end{aligned}$$

$$+\ \|F'(x_0)^{-1}F''(x_0)\| \cdot \|z - x_0\|$$
$$\leq \ell_3 \int_0^1 \|z - x_0\|^2 \theta_1 \mathrm{d}\theta_1 + \ell_0 \|z - x_0\|$$
$$\leq \tfrac{\ell_3}{2}\|z - x_0\|^2 + \ell_0\|z - x_0\|.$$

Set $L = \int_0^1 F'(y^* + \theta(x^* - y^*))\mathrm{d}\theta$. Then we have for $z = y^* + \theta(x^* - y^*)$, $\theta \in [0,1]$:

$$\|z-x_0\| \leq (1-\theta)\|y^*-x_0\|+\theta\|x^*-x_0\| < (1-\theta)R_1+\theta v^* \leq (1-\theta)R_1+\theta R_1 = R_1.$$

Hence, we get

$$\begin{aligned} \|F'(x_0)^{-1}[L - F'(x_0)]\| &\leq \int_0^1 \left[\tfrac{\ell_3}{2}\|z - x_0\|^2 + \ell_0\|z - x_0\|\right] \mathrm{d}\theta \\ &< \tfrac{\ell_3}{8}(R_1 + v^*)^2 + \tfrac{\ell_0}{2}(R_1 + v^*) = \gamma \in [0,1]. \end{aligned}$$

By the above and the Banach Lemma on invertible operators L is invertible.

Using the identity

$$F(x^*) - F(y^*) = L(x^* - y^*),$$

we get

$$x^* = y^*.$$

That completes the proof of Theorem 5. □

Theorem 6. *Let $F\colon D \subseteq X \to Y$ be a twice Fréchet-differentiable operator. Assume: there exist a point $x_0 \in D$ and non-negative parameters η, ℓ_0, ℓ_1, ℓ_3, ℓ_4 such that*

$$\begin{aligned} F'(x_0)^{-1} &\in L(Y,X), \\ \|F'(x_0)^{-1}F(x_0)\| &\leq \eta, \\ \|F'(x_0)^{-1}F''(x_0)\| &\leq \ell_0, \\ \|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| &\leq \ell_1\|x - x_0\|, \\ \|F'(x_0)^{-1}[F''(x) - F''(x_0)]\| &\leq \ell_3\|x - x_0\| \end{aligned}$$

and

$$\|F'(x_0)^{-1}[F''(x) - F''(y)]\| \leq \ell_4\|x - y\| \quad \textit{for all } x, y \in D.$$

Moreover, hypotheses of Theorem 1 hold, and

$$\overline{U}(x_0, v^*) \subseteq D.$$

Then the method of tangent parabolas $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $\overline{U}(x_0, v^*)$ for all $n \geq 0$ and converges to a solution $x^* \in \overline{U}(x_0, v^*)$ of equation $F(x) = 0$. Moreover the following error bounds hold for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq v_{n+1} - v_n,$$

and

$$\|x_n - x^*\| \leq v^* - v_n.$$

Furthermore, if there exists $R_2 \geq v^*$ such that

$$U(x_0, R_2) \subseteq D,$$

and

$$\ell_1(v^* + R_2) \leq 2,$$

the solution x^* is unique in $U(x_0, R_2)$.

Proof. Use (23) instead of (31). The rest follows as in Theorem 5 until the uniqueness part. Moreover the uniqueness part follows as in Theorem 4.

That completes the proof of Theorem 6. □

REMARK 1. In order for us to compare our results with earlier ones in [1], [6], [12] define sequences $\{\delta_n\}$, $\{M_n\}$, $\{N_n\}$, $\{\beta_n\}$ by

$$\begin{aligned}
&(32) & \delta_0 &= \eta, \quad M_0 = \ell_0, \quad N_0 = \ell_4,\\
&(33) & h_n &= M_n\delta_n, \quad \tau_n = 1 + \tfrac{1}{2}h_n, \quad \beta_n = \tau_n\delta_n, \quad \rho_n = M_n\beta_n,\\
&(34) & \varepsilon_n &= \tfrac{N_n}{M_n^2}, \quad \phi_n(\rho) = \rho + \tfrac{1}{2}\varepsilon_n\rho^2,\\
&(35) & c_n^2 &= \left(\tfrac{1}{6}\varepsilon_n\tau_n^3 + \tfrac{1}{2} + \tfrac{1}{8}h_n\right)/(1 - \varphi_n(\rho_n)),\\
&(36) & k_n^2 &= \phi_n'(\rho_n)c_n^2h_n^2/(1 - \varphi_n(\rho_n)),\\
&(37) & \delta_{n+1} &= c_n^2h_n^2\delta_n, \quad M_{n+1} = M_n\phi_h'(\rho_n)/(1 - \phi_n(\rho_n)),\\
&(38) & N_{n+1} &= \tfrac{N_n}{1-\varphi_n(\rho_n)},\\
&(39) & w_0 &= 0, \quad w_{n+1} = w_n - \left[1 + \tfrac{f''(w_n)f(w_n)}{2f'(w_n)^2}\right]\tfrac{f(w_n)}{f'(w_n)} \quad (n \geq 0),
\end{aligned}$$

and function

$$f(t) = \tfrac{1}{6}\ell_4t^3 + \tfrac{1}{2}\ell_0t^2 - t + \eta. \tag{40}$$

Assume:

$$\begin{aligned}
\delta_0M_0 &< 2,\\
\phi_0(\rho_0) &< 1,\\
k_0^2 &\leq 1,\\
\overline{U}(x_0, r) &\subseteq D, \quad r = \tfrac{\beta_0}{1-c_0^2\rho_0^2},
\end{aligned}$$

or equation

$$f(t) = 0$$

has one negative and two positive roots w^*, w^{**} such that $w^* \leq w^{**}$ and $\overline{U}(x_0, w^*) \subseteq D$ or, equivalently,

$$\eta \leq \frac{\ell_0^2+4\ell_4-\ell_0\sqrt{\ell_0^2+2\ell_4}}{3\ell_4(\ell_0+\sqrt{\ell_0^2+2\ell_3})}, \ \ell_4 \neq 0 \ \ (\ell_0\eta \leq \tfrac{1}{2} \text{ for } \ell_4 = 0) \tag{41}$$

and

$$\overline{U}(x_0, w^*) \subseteq D. \tag{42}$$

Then, the method of tangent parabolas $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $\overline{U}(x_0, w^*)$ for all $n \geq 0$ and converges to a solution $x^* \in \overline{U}(x_0, w^*)$ of equation $F(x) = 0$. Moreover the following error bounds hold for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq w_{n+1} - w_n$$

and

$$\|x_n - x^*\| \leq w^* - w_n.$$

Furthermore, if: $w^* < w^{**}$ the solution is unique in $U(x_0, w^{**})$ otherwise the solution is unique in $\overline{U}(x_0, w^*)$.

In general we have:

$$\ell_3 \leq \ell_4. \tag{43}$$

If strict inequality holds in (43) using induction on n we can easily show under the hypotheses of Theorem 5 and (12)–(15), (19), (32)–(40), (41) and (42)

$$\begin{aligned} s_{n+1} - s_n &< w_{n+1} - w_n \quad (n \geq 1) \\ s_n &< w_n \quad (n \geq 1) \end{aligned}$$

and

$$s^* \leq w^*.$$

That is, our Theorem 5 provides more precise error bounds and a better information on the location of the solution x^*. In the case of $\ell_3 = \ell_4$, Theorem 5 reduces to earlier ones mentioned in this remark. □

We complete this study with two simple examples:

Example 1. Let $\ell_0 = \ell_3 = 0$, $\eta = 1$ and $\ell_4 = 1$. Then, (41) is violated since

$$\eta = 1 > \tfrac{\sqrt{2}}{2}.$$

Hence the results in [12] cannot be used. However all hypotheses of Theorems 2, 5 are satisfied, since $\alpha_0 = 1$, and (10) and (11) hold. □

In the next example we show that $\frac{\ell_4}{\ell_3}$ may be arbitrarily large.

Example 2. Let $X = Y = \mathbb{R}$, $x_0 = 0$ and define function F on $\mathbb{R}$ by

$$F(x) = \int_0^x G(t)\mathrm{d}t, \tag{44}$$

where

$$G(t) = c_0 t + c_1 + c_2 \sin \mathrm{e}^{c_3 t}, \tag{45}$$

where c_i, $i = 0, 1, 2, 3$ are given parameters. Using (44) and (45) we can easily see that for c_3 large and c_2 sufficiently small, $\frac{\ell_4}{\ell_3}$ may be arbitrarily large. That is (43) holds as strict inequality and (41) may be violated whereas hypotheses of Theorem 5 may hold (see also Example 1). □

REFERENCES

[1] Argyros, I.K., *On the convergence of a Chebysheff–Halley-type method under Newton–Kantorovich hypotheses*, Appl. Math. Letters, **6**, no. 5, pp. 71–74, 1993.

[2] Argyros, I.K., *A note on the Halley method in Banach spaces*, Appl. Math. and Comp., **58**, pp. 215–224, 1993.

[3] Argyros, I.K., *Advances in the Efficiency of Computational Methods and Applications*, World Scientific Publ. Co., River Edge, NJ, 2000.

[4] Ezquerro, J.A. and Hernandez, M.A., *A modification of the super-Halley method under mild differentiability conditions*, J. Comput. Appl. Math., **114**, pp. 405–409, 2000.

[5] Gutiérrez, J.M. and Hernandez, M.A., *A family of Chebyshev–Halley type methods in Banach spaces*, Bull. Austral. Math. Soc., **55**, pp. 113–130, 1997.

[6] Kanno, S., *Convergence theorems for the method of tangent hyperbolas*, Math. Japonica, **37** no. 4, pp. 711–722, 1992.

[7] Kantorovich, L.V. and Akilov, G.P., *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1982.

[8] Mertvecova, M.A., *An analog of the process of tangent hyperbolas for general functional equations*, Dokl. Akad. Nauk SSSR, **88**, pp. 611–614, 1953.

[9] Necepurenko, M.T., *On Chebysheff's method for functional equations*, Usephi, Mat. Nauk., **9**, pp. 163–170, 1954.

[10] Safiev, R.A., *The method of tangent hyperbolas*, Sov. Math. Dokl., **4**, pp. 482–485, 1963.

[11] Schwetlick, H., *Numerische Losung Nichtlinearer Gleichungen*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1979.

[12] Yamamoto, T., *On the method of tangent hyperbolas in Banach spaces*, J. Comput. Appl. Math., **21**, pp. 75–86, 1988.

Received by the editors: March 20, 2003.