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NEW TECHNIQUE FOR PROVING THE EQUIVALENCE OF MANN AND ISHIKAWA ITERATIONS

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Abstract. We show the equivalence bewteen the convergences of Mann and Ishikawa iterations dealing with various classes of non-Lipschitzian operators.

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1. INTRODUCTION

Let X be a real Banach space, B be a nonempty, convex subset of X, and $T : B \to B$ be an operator. Let $u_0, x_0 \in B$. The following iteration is known as Mann iteration, see [2]:

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n. \tag{1}$$

The Ishikawa iteration is given by, see [1]:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n. \end{aligned} \tag{2}$$

The sequences $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfy

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty. \tag{3}$$

The map $J : X \to 2^{X^*}$ given by $Jx := \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\}, \forall x \in X$, is called *the normalized duality mapping.* It is easy to see that we have

$$\langle y, j(x)\rangle \leq \|x\| \, \|y\|, \quad \forall x, y \in X, \forall j(x) \in J(x). \tag{4}$$

Denote

$$\Psi := \{\psi \mid \psi : [0,+\infty) \to [0,+\infty) \text{ is a nondecreasing map such that } \psi(0) = 0\}.$$

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Definition 1. *Let X be a real Banach space. Let B be a nonempty subset of X. A map $T : B \to B$ is called uniformly pseudocontractive if there exist map $\psi \in \Psi$ and $j(x-y) \in J(x-y)$ such that*

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \psi(\|x-y\|), \quad \forall x, y \in B. \tag{5}$$

The map $S : X \to X$ is called uniformly accretive if there exist map $\psi \in \Psi$ and $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y)\rangle \geq \psi(\|x-y\|), \quad \forall x, y \in X. \tag{6}$$

Taking $\psi(a) := \psi(a) \cdot a, \forall a \in [0, +\infty)$, $(\psi \in \Psi)$, we get the usual definitions of ψ-strongly pseudocontractivity and ψ-strongly accretivity. Taking $\psi(a) := \gamma \cdot a^2, \gamma \in (0,1)$, $\forall a \in [0, +\infty)$, $(\psi \in \Psi)$, we get the usual definitions of strong pseudocontractivity and strong accretivity. If $\gamma := 0$, then we get the definition of a pseudocontractive and accretive map.

The following inequality was used in [5]:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in X, \ \forall j(x+y) \in J(x+y). \tag{7}$$

We shall give new proofs for the results from [5]. Our technique is new and does not need inequality (7). Remark that inequality (7) is used in almost all recent results concerning Mann or Ishikawa iterations.

Lemma 2. *Let $\{a_n\}$ be a nonnegative bounded sequence which satisfies the following inequality*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n a_{n+1} - \alpha_n \frac{\psi(a_{n+1})}{a_{n+1}} + \alpha_n \varepsilon_n, \quad \forall n \geq n_0, \tag{8}$$

where $\alpha_n \in (0,1)$, $\varepsilon_n \geq 0$, $\forall n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\lim_{n\to\infty} \varepsilon_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.

Proof. There exists $m > 0$ such that $a_n \leq M, \forall n \in \mathbb{N}$. Denote $a := \liminf a_n$. We shall prove that $a = 0$. Else $a > 0$. Thus, there exists $N_1 \in \mathbb{N}$ such that

$$a_n \geq \frac{a}{2}, \quad \forall n \geq N_1.$$

Because $\lim_{n\to\infty} \varepsilon_n = 0$, there exists $N_2 \in \mathbb{N}$ such that

$$\varepsilon_n \leq \frac{\psi\left(\frac{a}{2}\right)}{2m}, \quad \forall n \geq N_2.$$

Set $N_0 := \max\{N_1, N_2\}$. Using $-\frac{1}{m} \geq -\frac{1}{a_{n+1}}$ we get,

$$\begin{aligned} a_{n+1} &\leq (1-\alpha_n)a_n + \alpha_n a_{n+1} - \alpha_n \frac{\psi(a_{n+1})}{a_{n+1}} + \alpha_n \varepsilon_n \\ &\leq (1-\alpha_n)a_n + \alpha_n a_{n+1} - \alpha_n \frac{\psi\left(\frac{a}{2}\right)}{m} + \alpha_n \frac{\psi\left(\frac{a}{2}\right)}{2m} \\ &\leq (1-\alpha_n)a_n + \alpha_n a_{n+1} - \alpha_n \frac{\psi\left(\frac{a}{2}\right)}{2m}. \end{aligned}$$

From which we obtain $(1-\alpha_n)a_{n+1} \leq (1-\alpha_n)a_n - \alpha_n \frac{\psi\left(\frac{a}{2}\right)}{2m}$, that is:

$$a_{n+1} \leq a_n - \frac{\alpha_n}{1-\alpha_n} \frac{\psi\left(\frac{a}{2}\right)}{2m} \leq a_n - \alpha_n \frac{\psi\left(\frac{a}{2}\right)}{2m},$$

because $-\frac{\alpha_n}{1-\alpha_n} \leq -\alpha_n$. Thus, we have $\alpha_n \frac{\psi\left(\frac{a}{2}\right)}{2m} \leq a_n - a_{n+1}$, which implies $\sum \alpha_n < \infty$, in contradiction with $\sum \alpha_n = \infty$. We proved that $\liminf a_n = 0$. Hence, there exists a subsequence $\{a_{n_j}\} \subset \{a_n\}$ such that $\lim_{j\to\infty} a_{n_j} = 0$. Take $\varepsilon > 0$, there exists $n_3 \in \mathbb{N}$ such that

$$a_{n_j} < \frac{\varepsilon}{4}, \quad \forall j \geq n_3.$$

Also, there exists $n_4 \in \mathbb{N}$ such that

$$\varepsilon_n < \frac{\psi\left(\frac{\varepsilon}{4}\right)}{2m}, \quad \forall n \geq n_4.$$

Set $n_0 := \max\{n_3, n_4, N_0\}$. We have $a_{n_j+k} < \frac{\varepsilon}{4}, \forall k > 0$. Else, $a_{n_j+k} \geq \frac{\varepsilon}{4}$. The following inequalities are satisfied.

$$\begin{aligned} a_{n_j+1} &\leq (1-\alpha_{n_j})a_{n_j} + \alpha_{n_j}a_{n_j+1} - \alpha_{n_j}\frac{\psi(a_{n_j+1})}{a_{n_j+1}} + \alpha_{n_j}\varepsilon_{n_j} \\ &\leq (1-\alpha_{n_j})a_{n_j} + \alpha_{n_j}a_{n_j+1} - \alpha_{n_j}\frac{\psi\left(\frac{\varepsilon}{4}\right)}{m} + \alpha_{n_j}\frac{\psi\left(\frac{\varepsilon}{4}\right)}{2m} \\ &\leq (1-\alpha_{n_j})a_{n_j} + \alpha_{n_j}a_{n_j+1} - \alpha_{n_j}\frac{\psi\left(\frac{\varepsilon}{4}\right)}{2m} \\ &\leq (1-\alpha_{n_j})\frac{\varepsilon}{4} + \alpha_{n_j}a_{n_j+1} - \alpha_{n_j}\frac{\psi\left(\frac{\varepsilon}{4}\right)}{2m}. \end{aligned}$$

From which we get $a_{n_j+1} \leq \frac{\varepsilon}{4} - \alpha_{n_j}\frac{\psi\left(\frac{\varepsilon}{4}\right)}{2m}$. This leads to the following contradiction,

$$\frac{\varepsilon}{4} \leq a_{n_j+1} \leq \frac{\varepsilon}{4} - \alpha_{n_j}\frac{\psi\left(\frac{\varepsilon}{4}\right)}{2m} < \frac{\varepsilon}{4}.$$

Therefore $a_{n_j+k} < \frac{\varepsilon}{4}, \forall k \in \mathbb{N}$ and hence $\lim_{n\to\infty} a_n = 0$. □

2. MAIN RESULT

Let x^* be a fixed point of T.

THEOREM 3. [5] *Let X be a real Banach space B be a nonempty, convex, bounded subset of X and let $T : B \to B$ be a uniformly continuous and uniformly pseudocontractive map. If $\{\alpha_n\}, \{\beta_n\}$ satisfy* (3)*, and $u_0 = x_0 \in B$, then the following are equivalent:*
(i) the Mann iteration (1) *converges to x^*,*
(ii) the Ishikawa iteration (2) *converges to x^*.*

Proof. The uniqueness of the fixed point comes from (5). Suppose that $\lim_{n\to\infty} u_n = x^*$. Using

$$\lim_{n\to\infty} \|x_n - u_n\| = 0, \tag{9}$$

and

$$0 \leq \|x^* - x_n\| \leq \|u_n - x^*\| + \|x_n - u_n\|$$

we get $\lim_{n\to\infty} x_n = x^*$. Conversely, suppose $\lim_{n\to\infty} x_n = x^*$, then

$$0 \leq \|x^* - u_n\| \leq \|x_n - x^*\| + \|x_n - u_n\| \to 0,$$

leads to $\lim_{n\to\infty} x_n = x^*$. The proof is complete if we prove the relation (9) .

Using (1), (2) and (4) we get

$$\begin{aligned}
&\|x_{n+1}-u_{n+1}\|^2 = \\
&= \langle x_{n+1}-u_{n+1}, j(x_{n+1}-u_{n+1})\rangle \\
&= \big\langle (1-\alpha_n)(x_n-u_n)+\alpha_n(Ty_n-Tu_n), j(x_{n+1}-u_{n+1})\big\rangle \\
&= (1-\alpha_n)\big\langle (x_n-u_n), j(x_{n+1}-u_{n+1})\big\rangle + \alpha_n\big\langle Ty_n-Tu_n, j(x_{n+1}-u_{n+1})\big\rangle \\
&\leq (1-\alpha_n)\|x_n-u_n\|\,\|x_{n+1}-u_{n+1}\| \\
&\quad + \alpha_n\big\langle Tx_{n+1}-Tu_{n+1}, j(x_{n+1}-u_{n+1})\big\rangle \\
&\quad + \alpha_n\big\langle Ty_n-Tx_{n+1}, j(x_{n+1}-u_{n+1})\big\rangle \\
&\quad + \alpha_n\big\langle Tu_{n+1}-Tu_n, j(x_{n+1}-u_{n+1})\big\rangle \\
&\leq (1-\alpha_n)\|x_n-u_n\|\,\|x_{n+1}-u_{n+1}\| \\
&\quad + \alpha_n\|x_{n+1}-u_{n+1}\|^2 - \alpha_n\psi(\|x_{n+1}-u_{n+1}\|) \\
&\quad + \alpha_n\|Ty_n-Tx_{n+1}\|\,\|x_{n+1}-u_{n+1}\| \\
&\quad + \alpha_n\|Tu_{n+1}-Tu_n\|\,\|x_{n+1}-u_{n+1}\| \\
&\leq \|x_{n+1}-u_{n+1}\|\big((1-\alpha_n)\|x_n-u_n\| \\
&\quad + \alpha_n\|x_{n+1}-u_{n+1}\| - \alpha_n\tfrac{\psi(\|x_{n+1}-u_{n+1}\|)}{\|x_{n+1}-u_{n+1}\|} \\
&\quad + \alpha_n\|Ty_n-Tx_{n+1}\| + \alpha_n\|Tu_{n+1}-Tu_n\|\big).
\end{aligned}$$

Suppose that $\|x_{n+1}-u_{n+1}\|=0$, then $\|x_{n+k}-u_{n+k}\|=0$, for all $k\geq 1$. We get our conclusion. Suppose now that $\|x_n-u_n\|\neq 0$, for all $n\in\mathbb{N}$. The following relation is satisfied.

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\| \leq{}& (1-\alpha_n)\|x_n-u_n\| + \alpha_n\|x_{n+1}-u_{n+1}\| \\
&- \alpha_n\tfrac{\psi(\|x_{n+1}-u_{n+1}\|)}{\|x_{n+1}-u_{n+1}\|} + \alpha_n\|Ty_n-Tx_{n+1}\| \\
&+ \alpha_n\|Tu_{n+1}-Tu_n\|.
\end{aligned}$$

We prove now that

$$\lim_{n\to\infty}\|Ty_n-Tx_{n+1}\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|Tu_{n+1}-Tu_n\|=0. \tag{10}$$

For (10) it is sufficient to see that

$$\begin{aligned}
\|x_{n+1}-y_n\| ={}& \|-\alpha_n x_n+\alpha_n Ty_n+\beta_n x_n-\beta_n Tx_n\| \\
\leq{}& \alpha_n(\|x_n\|+\|Ty_n\|)+\beta_n(\|x_n\|+\|Tx_n\|) \\
\leq{}& (\alpha_n+\beta_n)M\to 0 \quad (n\to\infty),
\end{aligned} \tag{11}$$

$$\|u_{n+1}-u_n\|\leq\alpha_n\|u_n-Tu_n\|\to 0 \quad (n\to\infty), \tag{12}$$

where $M=\sup_n(\|u_n\|+\|Tu_n\|,\|x_n\|+\|Ty_n\|,\|x_n\|+\|Tx_n\|)$. The sequences $\{x_n\}$, $\{Tx_n\}$, $\{Tu_n\}$ and $\{Ty_n\}$ are bounded being in the bounded set B. Hence, the $M>0$, above is finite and (10) holds.

Observe that $\lim_{n\to\infty}\|x_{n+1}-y_n\|=0$, $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$ and the uniform continuity of T lead to (10). Denote by $a_n:=\|x_n-u_n\|$ and use Lemma 2, to obtain $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\|x_n-u_n\|=0$. □

REMARK 1. If B is not bounded then Theorem 3 holds supposing only $\{x_n\}$ is bounded. If $T(B)$ is bounded then $\{x_n\}$ is bounded. □

3. FURTHER RESULTS

Let I denote the identity map.

REMARK 2. The operator T is a (uniformly, strongly) pseudocontractive map if and only if $(I-T)$ is a (uniformly, strongly) accretive map. □

REMARK 3. (1) Let $T,S:X\to X$, and $f\in X$ be given. A fixed point for the map $Tx=f+(I-S)\,x, \forall x\in X$ is a solution for $Sx=f$.
(2) Let $f\in X$ be a given point. If S is an accretive map then $T=f-S$ is a strongly pseudocontractive map. □

Consider Mann and Ishikawa iterations with $Tx=f+(I-S)x$:

$$\begin{aligned}x_{n+1}&=(1-\alpha_n)x_n+\alpha_n\left(f+(I-S)y_n\right),\\ y_n&=(1-\beta_n)x_n+\beta_n\left(f+(I-S)x_n\right),\end{aligned}\tag{13}$$

and

$$u_{n+1}=(1-\alpha_n)u_n+\alpha_n\left(f+(I-S)u_n\right).\tag{14}$$

Remarks 2 and 3 and Theorem 3 lead to the following results.

COROLLARY 4. [5] *Let X be a real Banach space and $S:X\to X$ a uniformly continuous and uniformly accretive map with $(I-S)(X)$ bounded. If $\{\alpha_n\},\{\beta_n\}$ satisfy* (3)*, and $u_0=x_0\in X$, then the following are equivalent:*

(i) *the Mann iteration* (14) *converges to a solution of $Sx=f$,*
(ii) *the Ishikawa iteration* (13) *converges to a solution of $Sx=f$.*

Let S be an accretive operator. The operator $Tx=f-Sx$ is strongly pseudocontractive, for a given $f\in X$. A solution for $Tx=x$ becomes a solution for $x+Sx=f$. Consider Mann and Ishikawa iterations with $Tx:=f-Sx$:

$$\begin{aligned}x_{n+1}&=(1-\alpha_n)x_n+\alpha_n\left(f-Sy_n\right),\\ y_n&=(1-\beta_n)x_n+\beta_n\left(f-Sx_n\right),\end{aligned}\tag{15}$$

and

$$u_{n+1}=(1-\alpha_n)u_n+\alpha_n\left(f-Su_n\right).\tag{16}$$

Again, using the Remarks 2 and 3 and Theorem 3 we obtain the following results which generalizes Corollary 3.2 from [3].

Corollary 5. [5] *Let X be a real Banach space and B a nonempty, convex, closed subset of X. Let $S : B \to B$ be an uniformly continuous and accretive operator with $(I - S)(X)$ bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy* (3)*. Then for $u_0 = x_0 \in B$ the following assertions are equivalent:*

(i) *the Mann iteration* (16) *converges to the solution of $x + Sx = f$;*

(ii) *the Ishikawa iteration* (15) *converges to the solution of $x + Sx = f$.*

In [4] we deal with the Lipschitzian version of the above results. Thus, in this paper we also generalized all the results from [4].

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