REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 34 (2005) no. 1, pp. 109–113
ictp.acad.ro/jnaat

ON SOME INEQUALITIES FOR THE APPROXIMATION NUMBERS OF THE SUM AND PRODUCT OF OPERATORS

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Abstract. We prove the inequalities:

$$\sum_{n=1}^{k} a_n\left(\sum_{i=1}^{r} S_i\right) \leq r \sum_{n=1}^{k} \sum_{i=1}^{r} a_n(S_i),$$

$$\sum_{n=1}^{k} a_n\left(\prod_{i=1}^{r} S_i\right) \leq r \sum_{n=1}^{k} \prod_{i=1}^{r} a_n(S_i)\ ,\ k = 1, 2, ...,\quad r \geq 2,$$

and

$$\prod_{n=1}^{k} a_n\left(\prod_{i=1}^{r} S_i\right) \leq \prod_{n=1}^{k} \prod_{i=1}^{r} a_n^r(S_i)\ ,\ k = 1, 2, ...,\quad r \geq 2,$$

where $\{a_n(S)\}$ is the sequence of the approximation numbers of the linear and bounded operators $S : X \to X$ $(S \in L(X))$. X is a Banach space.

MSC 2000. 47B06, 47B10.

Keywords. Approximation numbers, symmetric norming function.

1. INTRODUCTION

Let H be a Hilbert space and let $S : H \to H, S \in L(H)$, be a compact linear operator. We denote by $s_n(S) = \lambda_n(SS^*)^{\frac{1}{2}}$, where $\{\lambda_n(SS^*)^{\frac{1}{2}}\}$ is the sequence of the eigenvalues of $(SS^*)^{\frac{1}{2}}$, ordered decreasing and repeating each one as many times as its algebraic multiplicity.

The following inequalities of K. Fan [1], [2] and A. Horn [2], [3] are well known:

$$\sum_{n=1}^{k} s_n\left(S_1 + S_2\right) \leq \sum_{n=1}^{k} \left(s_n(S_1) + s_n(S_2)\right)\ ,\ k = 1, 2, \ldots, \tag{1}$$

$$\sum_{n=1}^{k} s_n(S_1S_2) \leq \sum_{n=1}^{k} (s_n(S_1)s_n(S_2)\ ,\ k = 1, 2, \ldots. \tag{2}$$

In [2] is proved that $s_n(S) = \inf\{\| S - A \|: A \in L(H),\ \text{rank}\ A < n\}$ (see also [6] , [9]).

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In this way we can consider X a Banach space and $S \in L(X)$. Then, the approximation numbers of $S, (a_n(S))$, are defined as follows:

$$a_n(S) := \inf \{\| S - A \|: A \in L(X), \text{ rank } A < n\}, \ n = 1, 2, \ldots.$$

It is known that $\| S \| = a_1(s) \geq a_2(s) \geq ... \geq 0$ and

$$(a)\ a_{m+n-1}(S_1 + S_2) \leq a_m(S_1) + a_n(S_2),\ m, n = 1, 2, \ldots,$$
$$(b)\ a_{m+n-1}(S_1 S_2) \leq a_m(S_1) a_n(S_2),\ m, n = 1, 2, \ldots.$$

By using the inequalities (a) and (b), for $m = n$, in [6], are obtained some inequalities of the above type (1) and (2), but in the right side appear the factor 2, because

$$\begin{aligned}\sum_{n=1}^{k} a_n(S_1 S_2) &\leq \sum_{n=1}^{2k} a_n(S_1 S_2)\\ &= \sum_{n=1}^{k} a_{2n-1}(S_1 S_2) + \sum_{n=1}^{k} a_{2n}(S_1 S_2)\\ &\leq 2 \sum_{n=1}^{k} a_{2n-1}(S_1 S_2)\\ &\leq 2 \sum_{1}^{k} a_n(S_1) a_n(S_2).\end{aligned}$$

By reiteration we obtain:

$$\sum_{n=1}^{k} a_n \left(\prod_{i=1}^{r} S_i \right) \leq 2^{r-1} \sum_{n=1}^{k} \prod_{i=1}^{r} a_n(S_i), \quad k = 1, 2, ..., \quad r \geq 2.$$

An analog inequality is true for the sum of r operators $(S_i \in L(X))$.

The purpose of this paper is to prove, in a simple way, that the factor 2^{r-1} can be replaced by r. Some applications are also presented.

2. RESULTS

Firstly we prove a generalization of the inequalities (1) and (2), for $m = n$.

LEMMA 1. *The approximation numbers verify the following inequalities:*

(3) $$a_{(n-1)r+1}(S_1 + ... + S_r) \leq a_n(S_1) + ... + a_n(S_r), \ \ n = 1, 2, ..., \ r \geq 2$$

(4) $$a_{(n-1)r+1}(S_1 ... S_r) \leq a_n(S_1) ... a_n(S_r), \ \ n = 1, 2, ..., \ r \geq 2.$$

Proof. We prove (4) by induction relative to $r \geq 2$. For $r = 2$, we obtain $a_{2n-1}(S_1 S_2) \leq a_n(S_1) a_n(S_2)$ which is (b), for $m = n$.

We suppose that $a_{(n-1)(r-1)+1}(S_1 ... S_{r-1}) = a_{n(r-1)-(r-2)}(S_1 ... S_{r-1}) \leq a_n(S_1) ... a_n(S_{r-1})$, i.e. (4) is true for $r - 1$ and we prove for r.

Then, by (2), we obtain:

$$\begin{aligned} a_{(n-1)r+1}(S_1...S_r) &= a_{nr-(r-1)}(S_1...S_r) \\ &= a_{n(r-1)+n-(r-2)-1}(S_1...S_r) \\ &\le a_{n(r-1)-(r-2)}(S_1...S_{r-1}) \cdot a_n(S_r) \\ &\le a_n(S_1)...a_n(S_r). \end{aligned}$$

The inequality (3) results in a same way by using the inequality (a). □

By means of these inequalities we obtain:

THEOREM 2. *For the operators $S_i \in L(X)$, $i = 1, 2, ...r$, the following inequalities hold:*

$$\sum_{n=1}^{k} a_n\left(\sum_{i=1}^{r} S_i\right) \le r\sum_{n=1}^{k}\sum_{i=1}^{r} a_n(S_i)\ ,\ k = 1, 2, ...,\ r \ge 2, \tag{5}$$

$$\sum_{n=1}^{k} a_n\left(\sum_{i=1}^{r} S_i\right) \le r\sum_{n=1}^{k}\prod_{i=1}^{r} a_n(S_i)\ ,\ k = 1, 2, ...,\ r \ge 2, \tag{6}$$

$$\prod_{n=1}^{k} a_n\left(\prod_{i=1}^{r} S_i\right) \le \prod_{n=1}^{k}\prod_{i=1}^{r} a_n^r(S_i)\ ,\ k = 1, 2, ...,\ r \ge 2. \tag{7}$$

Proof. We prove only (6) and (7) since the proof of (5) is similar.For to prove (6) we use the inequality (4) and the fact that the sequence $(a_n(S))$ is decreasing. Then we obtain:

$$\begin{aligned} \sum_{n=1}^{k} a_n(S_1...S_r) &\le \sum_{n=1}^{rk} a_n(S_1...S_r) \\ &= \sum_{n=1}^{k}\sum_{i=(n-1)r+1}^{i=nr} a_i\,(S_1...S_r) \\ &\le r\sum_{n=1}^{k} a_{(n-1)r+1}\,(S_1...S_r) \\ &\le r\sum_{n=1}^{k} a_n(S_1)...a_n(S_r). \end{aligned}$$

Also

$$\begin{aligned} \prod_{n=1}^{k} a_n\left(\prod_{i=1}^{r} S_i\right) &\le \prod_{n=1}^{rk} a_n\left(\prod_{i=1}^{r} S_i\right) \\ &= \prod_{n=1}^{k}\prod_{i=(n-1)r+1}^{nr} a_i\left(\prod_{i=1}^{r} S_i\right) \end{aligned}$$

$$\leq \prod_{n=1}^{r} a_{(n-1)r+1}^{r}\left(\prod_{i=1}^{r} S_i\right)$$
$$\leq \prod_{n=1}^{k}\prod_{i=1}^{r} a_n^r(S_i).$$

The proof is fulfiled. □

Remark 1. These inequalities are also true for all additive and multiplicative s-number sequences and also for the dyadic entropy numbers.

3. APPLICATION

Let l_∞ be the space of all bounded sequences

$$\left(x \in l_\infty \text{ if } \| x \|_\infty = \sup_n |x_n| < \infty\right).$$

For all $x \in l_\infty$, $\mathrm{card}(x) := \mathrm{card}\{i \in N : x_i \neq 0\}$. Let K be the set of all sequence $x = (x_n) \in l_\infty$ such that card $(x) \leq n$ and $x_1 \geq x_2 \geq ... \geq 0$. A function $\phi : K \to R$ is called symmetric norming function [2], [4]–[7], if

1. $\phi(x) > 0$, for $x \neq 0$;
2. $\phi(x+y) \leq \phi(x) + \phi(y), x, y \in K$;
3. $\phi(\alpha x) = \alpha\phi(x), \alpha \geq 0, x \in K$;
4. $\phi(1, 0, 0, ...) = 1$;
5. If $x, y \in K$ are such that

$$\sum_{i=1}^{r} x_i \leq \sum_{i=1}^{r} y_i,\ r = 1, 2...$$

then $\phi(x) \leq \phi(y)$.

In [2], [4], the classes of operators $L_\phi(H) = \{S \in L(H) : \phi(s_n(S)) < \infty\}$ are considered. By means of the properties of the function ϕ and the fact that

$$\sum_{1}^{k} s_n\left(S_1 + S_2\right) \leq \sum_{1}^{k}\left(s_n\left(S_1\right) + s_n(S_2)\right)$$

inequality (1) and $s_n(\alpha S) = |\alpha|\, s_n(S)$, it results that $L_\phi(H)$ is a normed space. The norm is $\|S\|_\phi = \phi(s_n(S))$. If $x \in l_\infty$ and $x_1 \geq x_2 \geq ... \geq 0$, we consider

$$\phi(x) = \lim_{n\to\infty} \phi(x_1, ..., x_n, 0, 0, ...).$$

In [6], [7], the classes $L_\phi(X)$, where X is a Banach space, are considered. $L_\phi(X) = \{S \in L(X) : \phi\left(a_n(S)\right) < \infty\}$. From the properties of the functions ϕ and the inequality (3) it results that $L_\phi(X)$ is a quasi normed space. The quasi-norm is $\|S\|_\phi = \phi(a_n(S))$. We remark that, from the inequality (3), we obtain

$$\|S_1 + S_2\|_\phi \leq 2(\|S_1\|_\phi + \|S_2\|_\phi).$$

By reiteration, from the above inequality we obtain [6], [9]:

$$\| \sum_{i=1}^{r} S_i \|_\phi \leq 2^{r-1} \left(\sum_{i=1}^{r} \| S_i \|_\phi \right), \ r \geq 2.$$

From the inequality (3), it results a very well relation:

$$\| \sum_{i=1}^{r} S_i \|_\phi \leq r \left(\sum_{i=1}^{r} \|S_i\|_\phi \right), \ r \geq 2,$$

The last relation is obtained as follows:

$$\begin{aligned} \| \sum_{i}^{r} S_i \|_\phi &= \phi \left(a_n \left(\sum_{i}^{r} S_i \right) \right) \\ &\leq r\phi \left(\sum_{i}^{r} a_n \left(S_i \right) \right) \\ &\leq r \left(\sum_{i=1}^{r} \phi \left(a_n(S_i) \right) \right) \\ &= r \sum_{i}^{r} \| S_i \|_\phi \ . \end{aligned}$$

In the case of Hilbert spaces, we obtain

$$\| \sum_{i}^{r} S_i \|_\phi \leq \sum_{1}^{r} \| S_i \|_\phi,$$

by using the inequality (1) and reiteration.

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Received by the editors: September 1, 2000.