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AN APPROXIMATION OPERATOR OF STANCU TYPE

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Abstract. We study the behavior of Stancu-Goldman's operator on the second degree functions.

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1. INTRODUCTION

Using the Pòlya-Eggenberger-Markov distribution, D. D. Stancu has defined in [4] a polynomial approximation operator, which today is bearing his name (see [1]). The above distribution was generalized by B. Friedman in [2]. Using this new distribution, R. N. Goldman has defined in [3] an approximation operator of Stancu type for which has proved that it preserves the first degree functions. In this paper, we begin the study of the behavior of this operator on the second degree functions. This is necessary for the application of Popoviciu-Bohman-Korovkin approximation theorem (see [1]).

2. FRIEDMAN'S DISTRIBUTION

In [2] it is considered the following probabilistic model: an urn contains N balls, a of which are white and b black. A ball is draw out at random, its color noted and it is returned together with c balls of the same color and d balls of the opposite color. This procedure is repeated n times. We denote with $P(n,k)$ the probability that the total number of white balls chosen be k.

This model was studied in detail in [3] where the following recurrence relation

$$P(n,k) = P(n-1,k-1)\cdot\tfrac{a+(k-1)\cdot c+(n-k)\cdot d}{N+(n-1)\cdot(c+d)} + P(n-1,k)\cdot\tfrac{b+(n-k-1)\cdot c+k\cdot d}{N+(n-1)\cdot(c+d)}$$

is established. If we denote

$$\tfrac{a}{N}=x,\ \tfrac{c}{N}=\alpha,\ \tfrac{d}{N}=\beta$$

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and $P(n,k) = P(n,k,x)$ the recurrence relation becomes

$$P(n,k,x) = s(n-1,k-1,x) \cdot P(n-1,k-1,x) \\ + t(n-1,k,x) \cdot P(n-1,k,x),$$

where

$$s(n,k,x) = [x + k \cdot \alpha + (n-k) \cdot \beta] \cdot \rho_n$$

and

$$t(n,k,x) = 1 - s(n,k,x)$$

with

$$\rho_n = \tfrac{1}{1+n\cdot(\alpha+\beta)}.$$

For $\beta = 0$ we obtain the Pòlya-Eggenberger-Markov distribution with the explicit expression

$$P(n,k,x) = \tbinom{n}{k} \tfrac{x^{(k,-\alpha)}(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}},$$

where

$$x^{(k,-\alpha)} = x(x+\alpha)...[x+(k-1)\alpha].$$

As we have shown in [5], such an expression seems to be impossible to obtain in the case $\beta \neq 0$.

3. THE DETERMINATION OF THE MOMENTS

Though the expressions of the distribution are missing, as it is shown in [3] we can calculate its moments of order r

$$M_r(n,x) = \sum_{k=0}^{n} k^r \cdot P(n,k,x).$$

Of course

$$M_0(n,x) = 1,\ M_r(1,x) = x.$$

For the calculation of other moments, the following recurrence relation

$$M_r(n+1,x) = \sum_{i=0}^{r} M_i(n,x) \cdot \Gamma(r,n,x,i)$$

is used, where

$$\Gamma(r,n,x,i) = \Phi(r,n,i) \cdot x + \Psi(r,n,i)$$

with

$$\Phi(r,n,r) = 0,\ \Psi(r,n,r) = 1 + r \cdot (\alpha - \beta) \cdot \rho_n$$

and

$$\Phi(r,n,i) = \tbinom{r}{i} \cdot \rho_n,\ \Psi(r,n,i) = \left[\tbinom{r}{i-1} \cdot (\alpha-\beta) + \beta \cdot n \cdot \tbinom{r}{i}\right] \cdot \rho_n,\ i < r.$$

In the special case $r = 1$ we obtain:

$$M_1(n+1,x) = [1 + (\alpha-\beta) \cdot \rho_n] \cdot M_1(n,x) + (x + \beta \cdot n) \cdot \rho_n$$

which leads at

LEMMA 1. *There exist the constants* $p_n > 0,\ q_n \geq 0$, *such that*

(1) $$M_1(n,x) = p_n \cdot x + q_n,\ n \geq 1.$$

The proof is done by mathematical induction, starting with the initial values

$$p_1 = 1,\ q_1 = 0$$

obtaining the recurrence relations

$$p_{n+1} = \mu_n \cdot p_n + \rho_n > 0$$

and

$$q_{n+1} = \mu_n \cdot q_n + \beta \cdot n \cdot \rho_n \geq 0,$$

where

$$\mu_n = [1 + \alpha - \beta + (\alpha + \beta) \cdot n] \cdot \rho_n.$$

To get explicit expressions we use the following

LEMMA 2. *If the sequence* $(x_n)_{n \geq l}$ *verifies the recurrence relation*

$$x_{n+1} = A_n \cdot x_n + B_n,\ n \geq l$$

then it has the expression

$$x_n = \sum_{i=l}^{n-1} \left(B_i \prod_{j=i+1}^{n-1} A_j \right) + x_l \prod_{i=l}^{n-1} A_i,\ n > l$$

with the convention

$$\prod_{j=n}^{n-1} \cdots = 1.$$

We obtain

THEOREM 1. *The coefficients of the mean values* (1) *have the expressions*

$$p_n = \sum_{i=0}^{n-1} \left(\rho_i \prod_{j=i+1}^{n-1} \mu_j \right)$$

and

$$q_n = \beta \cdot \sum_{i=0}^{n-1} \left(i \cdot \rho_i \prod_{j=i+1}^{n-1} \mu_j \right).$$

Analogously, for the moments of order two we have

THEOREM 2. *There exist the positive constants* u_n, v_n, w_n *such that*

$$M_2(1,x) = x$$

and

(2) $$M_2(n,x) = u_n \cdot x^2 + v_n \cdot x + w_n,\ n \geq 2,$$

where

$$u_n = \sum_{i=1}^{n-1} \Big(2 \cdot \rho_i \cdot p_i \cdot \prod_{j=i+1}^{n-1} \varphi_j \Big),$$

$$v_n = \sum_{i=1}^{n-1} \Big[(2 \cdot \rho_i \cdot q_i + \psi_i \cdot p_i + \rho_i) \cdot \prod_{j=i+1}^{n-1} \varphi_j \Big] + \prod_{i=1}^{n-1} \varphi_i,$$

$$w_n = \sum_{i=1}^{n-1} \Big[(\psi_i \cdot q_i + \beta \cdot i \cdot \rho_i) \cdot \prod_{j=i+1}^{n-1} \varphi_j \Big],$$

with

$$\varphi_i = [1 + 2 \cdot (\alpha - \beta) + (\alpha + \beta) \cdot i] \cdot \rho_i$$

and

$$\psi_i = (\alpha - \beta + 2 \cdot \beta \cdot i) \cdot \rho_i.$$

Proof. For $r = 2$ the recurrence relation is

$$M_2(n+1, x) = \varphi_n \cdot M_2(n, x) + (2 \cdot \rho_n \cdot x + \psi_n) \cdot M(n, x) + \rho_n \cdot (x + \beta \cdot n).$$

So

$$M_2(2, x) = 2 \cdot \rho_1 \cdot x^2 + (\varphi_1 + \psi_1 + \rho_1) \cdot x + \beta \cdot \rho_1,$$

thus (2) is verified for $n = 2$. If we assume it to be valid for a given n, we deduce

$$\begin{aligned} M_2(n+1, x) = {} & (u_n \cdot x^2 + v_n \cdot x + w_n) \cdot \varphi_n \\ & + (p_n \cdot x + q_n) \cdot (2 \cdot \rho_n \cdot x + \psi_n) + \rho_n \cdot (x + \beta \cdot n). \end{aligned}$$

We have so (2) for $n + 1$, with

$$u_{n+1} = \varphi_n \cdot u_n + 2 \cdot \rho_n \cdot p_n,$$
$$v_{n+1} = \varphi_n \cdot v_n + 2 \cdot \rho_n \cdot q_n + \psi_n \cdot p_n + \rho_n$$

and

$$w_{n+1} = \varphi_n \cdot w_n + \psi_n \cdot q_n + \beta \cdot n \cdot \rho_n.$$

For ending the proof it is sufficient to apply the last lemma with the initial values

$$u_1 = 0,\ v_1 = 1 \text{ and } w_1 = 0. \qquad \square$$

4. THE DEFINITION OF THE APPROXIMATION OPERATOR

Using Friedman's distribution, the following operator

$$U_n : C[a, b] \to C[0, 1]$$

was defined in [3] by

$$(U_n f)(x) = \sum_{k=0}^{n} P(n, k, x) \cdot f(x_{n,k}), \tag{3}$$

where

$$a \leq x_{n,0} < x_{n,1} < ... < x_{n,n} \leq b.$$

This operator is linear, positive, of polynomial type, and with the property that

$$U_n e_0 = e_0,$$

where

$$e_k(x) = x^k,\ k = 0, 1, 2, ...$$

To apply Korovkin's approximation theorem (see [1]) the knots $x_{n,k}$ must be determined so that

$$U_n e_k \to e_k,\ n \to \infty,\ k = 1, 2.$$

As it is stated in [3], Ch. Micchelli had the idea of choosing the knots as follows.

THEOREM 3. *If the interval of definition of the functions verifies the condition*

$$[a, b] \supseteq \left[-\tfrac{q_n}{p_n}, \tfrac{n-q_n}{p_n}\right],$$

where p_n, q_n *are from* (1)*, then choosing the knots*

$$x_{n,k} = \tfrac{k-q_n}{p_n},\ k = 0, 1, ..., n,$$

the operator (3) *reproduces the linear functions such as it has the property*

$$U_n e_1 = e_1.$$

The proof is done by direct computation. In a similar way we obtain

THEOREM 4. *In the conditions of the above theorem, we have*

$$U_n e_2 = \tfrac{1}{p_n^2} \cdot \left[u_n \cdot e_2 + (v_n - 2 \cdot q_n \cdot p_n) \cdot e_1 + (w_n - q_n^2) \cdot e_0\right].$$

Proof. Step by step we have

$$\begin{aligned}(U_n e_2)(x) &= \sum_{k=0}^{n} \left(\tfrac{k-q_n}{p_n}\right)^2 \cdot P(n, k, x) \\ &= \tfrac{1}{p_n^2} \cdot \left[M_2(n, x) - 2 \cdot q_n \cdot M_1(n, x) + q_n^2\right] \\ &= \tfrac{1}{p_n^2} \cdot \left[u_n \cdot x^2 + v_n \cdot x + w_n - 2 \cdot q_n \cdot (p_n \cdot x + q_n) + q_n^2\right]\end{aligned}$$

which gives the desired result. □

5. A SPECIAL CASE

Choosing $\beta = 0$ we obtain Stancu's operator. Let us study the case $\alpha = 0$. We have first of all

$$\rho_n = \frac{1}{1+\beta\cdot n} \text{ and } \mu_n = \frac{1+\beta\cdot(n-1)}{1+\beta\cdot n}.$$

As

$$\prod_{j=i+1}^{n-1} \mu_j = \frac{1+\beta\cdot i}{1+\beta\cdot(n-1)}$$

it follows that

$$p_n = \frac{n}{1+\beta\cdot(n-1)} \text{ and } q_n = \frac{\beta\cdot n\cdot(n-1)}{2\cdot[1+\beta\cdot(n-1)]}.$$

Then

$$\varphi_n = \frac{1+\beta\cdot(n-2)}{1+\beta\cdot n} \text{ and } \psi_n = \frac{\beta\cdot(2\cdot n-1)}{1+\beta\cdot n}.$$

So

$$\prod_{j=i+1}^{n-1} \varphi_j = \frac{[1+\beta\cdot(i-1)]\cdot(1+\beta\cdot i)}{[1+\beta\cdot(n-2)]\cdot[1+\beta\cdot(n-1)]},$$

which gives

$$u_n = \frac{n\cdot(n-1)}{[1+\beta\cdot(n-2)]\cdot[1+\beta\cdot(n-1)]},$$

$$v_n = \frac{n\cdot[\beta\cdot n\cdot(n-2)+1]}{[1+\beta\cdot(n-2)]\cdot[1+\beta\cdot(n-1)]}$$

and

$$w_n = \frac{\beta\cdot n\cdot(n-1)\cdot[\beta\cdot(3\cdot n^2-5\cdot n-2)+6]}{12\cdot[1+\beta\cdot(n-2)]\cdot[1+\beta\cdot(n-1)]}.$$

We deduce that

$$\frac{u_n}{p_n^2} \to 1,\ n \to \infty,$$

and

$$\frac{v_n - 2\cdot p_n \cdot q_n}{p_n^2} \to 0,\ n \to \infty,$$

but

$$\frac{w_n - q_n^2}{p_n^2} \to \infty,\ n \to \infty,$$

so $U_n e_2$ does not converge.

It remains as an open problem that of determination of the parameters α and β such that

$$U_n e_2 \to e_2,\ n \to \infty.$$

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