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APPROXIMATION PROPERTIES OF A BIVARIATE STANCU TYPE OPERATOR

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Abstract. An extension of Stancu's operator $P_m^{(\alpha,\beta)}$ to the case of bivariate functions is presented and some approximation properties of this operator are discussed.

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1. PRELIMINARIES

In 1968 (see [8]), D.D. Stancu constructed and studied a linear and positive operator, depending on two positive parameters α and β which satisfy the condition $0 \leq \alpha \leq \beta$. This operator, denoted by $P_m^{(\alpha,\beta)}$, associates to any function $f \in C([0,1])$ the polynomial $P_m^{(\alpha,\beta)} f$, defined by:

$$(1) \qquad (P_m^{(\alpha,\beta)} f)(x) = \sum_{k=0}^{m} p_{mk}(x) \; f\left(\tfrac{k+\alpha}{m+\beta}\right),$$

where $p_{mk}(x)$ are the fundamental Bernstein polynomials. In the monograph by F. Allovave and M. Campiti [1] this operator is called "the operator of Bernstein Stancu".

A first extensions of the operator (1) to the case of bivariate functions was given by F. Stancu in her doctoral thesis (see [9]). The aim of the present paper is to extend the operator (1) to the case of B-continuous (Bőgel continuous functions). More exactly, we shall present a GBS (Generalized Boolean Sum) operator of Stancu type and some properties of this operator.

The terminology of "B-continuous function" was introduced by K. Bőgel [5], [6]. A first result concerning the approximation of this kind of functions is due to E. Dobrescu and I. Matei [7].

An important "test function theorem", (the analogous of the well known Korovkin theorem), for the approximation of B-continuous functions by GBS

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operators was introduced by C. Badea and C. Cottin [3]. Approximation properties of the GBS operators were studied C. Badea, C. Cottin, H.H. Gonska, D. Kacso and many others.

2. THE GBS OPERATOR OF STANCU TYPE

Let $I=[0,1]$ and let $I^2=[0,1]\times[0,1]$ be the unit square. The space of all B-continuous functions on I^2 will be denoted by $C_b(I^2)$.

Next, we consider four non-negative parameters $\alpha_1,\beta_1,\alpha_2,\beta_2$,satisfying the conditions $0\le\alpha_1\le\beta_1, 0\le\alpha_2\le\beta_2$. If $f\in C_b(I^2)$, the parametric extensions of the operator $P_m^{(\alpha,\beta)}$are defined respectively by:

$$\left({}_xP_m^{(\alpha_1,\beta_1)}f\right)(x,y)=\sum_{k=0}^{m}p_{mk}(x)f\left(\tfrac{k+\alpha_1}{m+\beta_1},y\right), \tag{2}$$

$$\left({}_yP_n^{(\alpha_2,\beta_2)}f\right)(x,y)=\sum_{l=0}^{n}p_{nl}(y)f\left(x,\tfrac{l+\alpha_2}{n+\beta_2}\right). \tag{3}$$

It is easy to see that ${}_xP_m^{(\alpha_1,\beta_1)}$and ${}_yP_n^{(\alpha_2,\beta_2)}$are linear and positive operators, well defined on $C_b(I^2)$.

Let $L_{m,n}: C_b(I^2)\to C_b(I^2)$ be the tensorial product of ${}_xP_m^{(\alpha_1,\beta_1)}$ and ${}_yP_n^{(\alpha_2,\beta_2)}$, i.e.

$$L_{m,n}={}_xP_{my}^{(\alpha_1,\beta_1)}\circ P_n^{(\alpha_2,\beta_2)}. \tag{4}$$

Then, $L_{m,n}: C_b(I^2)\to C_b(I^2)$ associates to any $f\in C_b(I^2)$ the bivariate polynomial

$$L_{m,n}\,f(x,y)=\sum_{k=0}^{m}\sum_{l=0}^{n}p_{mk}(x)p_{n,l}(y)f\left(\tfrac{k+\alpha_1}{m+\beta_1},\tfrac{l+\alpha_2}{n+\beta_2}\right). \tag{5}$$

It is well known (see for example [4] or [10]) that the operator (5) has the following properties:

LEMMA 1. *If $e_{ij}: I^2\to R$ $(i,j\in N, 0\le i+j\le 2)$ are the test functions, the following equalities hold*

(i) $(L_{m,n}e_{00})(x,y)=1$;
(ii) $(L_{m,n}e_{10})(x,y)=x+\frac{\alpha_1-\beta_1x}{m+\beta_1}$;
(iii) $(L_{m,n}e_{01})(x,y)=y+\frac{\alpha_2-\beta_2y}{n+\beta_2}$;
(iv) $(L_{m,n}e_{20})(x,y)=x^2+\frac{mx(1-x)+(\alpha_1-\beta_1x)(2mx+\beta_1x+\alpha_1)}{(m+\beta_1)^2}$;
(v) $(L_{m,n}e_{02})(x,y)=y^2+\frac{ny(1-y)+(\alpha_2-\beta_2y)(2ny+\beta_2y+\alpha_2)}{(m+\beta_2)^2}$;

for any $(x,y)\in I^2$.

LEMMA 2. *The operator (5) is linear and positive.*

Definition 1. *Let $S_{m,n} : C_b(I^2) \to C_b(I^2)$ be the boolean sum of ${}_xP_m^{(\alpha_1,\beta_1)}$ and ${}_yP_n^{(\alpha_2,\beta_2)}$, i.e.*

$$S_{m,n} = {}_x P_m^{(\alpha_1,\beta_1)} + {}_y P_n^{(\alpha_2,\beta_2)} - {}_x P_m^{(\alpha_1,\beta_1)} \circ {}_y P_n^{(\alpha_2,\beta_2)}. \tag{6}$$

The operator $S_{m,n}$ will be called GBS operator of Stancu type.

By direct computation, one obtains:

Lemma 3. *If $S_{m,n} : C_b(I^2) \to C_b(I^2)$ is the GBS operator of Stancu type, then*

(7)

$$(S_{m,n}f)(x,y) = \\ = \sum_{k=0}^{m}\sum_{l=0}^{n} p_{mk}(x)\, p_{nl}(y) \left\{ f\left(\tfrac{k+\alpha_1}{m+\beta_1}, y\right) + f\left(x, \tfrac{l+\alpha_2}{n+\beta_2}, y\right) - f\left(\tfrac{k+\alpha_1}{m+\beta_1}, \tfrac{l+\alpha_2}{n+\beta_2}\right)\right\}$$

for any $f \in C_b(I^2)$ and any $(x,y) \in I^2$.

Remark 1. For $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$, the GBS operator of Stancu type is reduced to the GBS operator of Bernstein type, which interpolates any function $f \in C_b(I^2)$ on the boundary of the unit square I^2. If $\alpha_1 = \beta_1 = 0$ and $\alpha_2 \neq 0, \beta_2 \neq 0$, the corresponding operator interpolates any $f \in C_b(I^2)$ on the left and respectively on the right side of the boundary of unit square I^2. Other particular cases of the GBS operator of Stancu type can be discussed in a similar way. □

Theorem 1. *For any $f \in C_b(I^2)$,the sequence $\{S_{m,n}f\}_{m,n\in\mathbb{N}}$ converges to f, uniformly on I^2 as m and n tend to infinity.*

Proof. Let us to introduce the following notations

$$\begin{aligned} u_m(x) &= \tfrac{\alpha_1-\beta_1 x}{m+\beta_1}, \\ v_n(y) &= \tfrac{\alpha_2-\beta_2 y}{n+\beta_2}, \\ w_m, n(x,y) &= x^2 + y^2 + \tfrac{mx(1-x)+(\alpha_1-\beta_1 x)(2mx+\beta_1+\alpha_1)}{(m+\beta_1)^2} \\ &\quad + \tfrac{ny(1-y)+(\alpha_2-\beta_2 y)(2ny+\beta_2+\alpha_2)}{(n+\beta_2)^2}. \end{aligned}$$

Then the results contained in Lemma 1 can be written in the form

$$\begin{aligned} (L_{m,n}e_{00})(x,y) &= 1; \\ (L_{m,n}e_{10})(x,y) &= x + u_m(x); \\ (L_{m,n}e_{01})(x,y) &= y + v_n(y); \\ (L_{m,n}(e_{20}+e_{02}))(x,y) &= x^2 + y^2 + w_{m,n}(x,y), \end{aligned}$$

for any $(x,y) \in I^2$.

Because the sequences $\{u_m(x)\}_{m\in\mathbb{N}}$, $\{v_n(x)\}_{n\in\mathbb{N}}$ and $\{w_{m,n}(x)\}_{m,n\in\mathbb{N}}$ tend to zero, uniformly on I^2 as m and n tend to infinity, we can apply the Korovkin-type theorem for the approximation of B-continuous functions due C. Badea, I. Badea and H.H. Gonska (see [2]). Applying this theorem, it follows that $S_{m,n}f$ tend to f, uniformly on I^2, for any $f \in C_b(I^2)$ as m and n tend to infinity. □

Next the approximation order of any function $f \in C_b(I^2)$ by $S_{m,n}f$ will be established, using the mixed modulus of smoothness (see [3]). We need the following result, due to C. Badea and C. Cottin (see [3]).

THEOREM 2. *Let X and Y be compact real intervals. Furthermore, let $L : C_b(X,Y) \to C_b(X,Y)$ be a positive linear operator and U the associated GBS operator. Then, for all $f \in C_b(X,Y)$, $(x,y) \in X \times Y$ and $\delta_1,\delta_2 > 0$ the inequality*

$$\begin{aligned}(8)\qquad & |(f - Uf)(x,y)| \le \\ & \le |f(x,y)| \cdot |1 - L(x;x,y| + \\ & + \{L(1;x,y) + \tfrac{1}{\delta_1}\sqrt{L((x-\circ)^2;x,y)} + \tfrac{1}{\delta_2}\sqrt{L((y-*)^2;x,y)} \\ & + \tfrac{1}{\delta_1\delta_2}\sqrt{L((x-\circ)^2(y-*)^2;x,y)}\ \}\,\omega_{mixed}(\delta_1,\delta_2)\end{aligned}$$

holds.

LEMMA 4. *The bivariate operator of Stancu verifies the following equalities:*

$$\begin{aligned} L_{m,n}((x-\circ)^2;x,y) &= \tfrac{mx(1-x)+(\alpha_1-\beta_1 x)^2}{(m+\beta_1)^2}; \\ L_{m,n}((y-*)^2;x,y) &= \tfrac{ny(1-y)+(\alpha_2-\beta_2 y)^2}{(n+\beta_2)^2}; \\ L_{m,n}((x-\circ)^2(y-*)^2 &= \tfrac{1}{(m+\beta_1)^2(n+\beta_2)^2}\left\{mx(1-x)+(\alpha_1-\beta_1 x)^2\right\} \\ & \times\left\{ny(1-y)+(\alpha_2-\beta_2 y)^2\right\}. \end{aligned}$$

Proof. The equalities follow from the linearity of L_{mn} and Lemma 1. □

THEOREM 3. *The GBS operators of Stancu S_{mn} verifies the inequality:*

$$\begin{aligned}(9)\qquad & |S_{m,n}f(x,y) - f(x,y)| \le \\ & \le \left\{\tfrac{1}{\delta 1}\cdot\tfrac{1}{m+\beta_1}\sqrt{\tfrac{m}{4}+(\alpha_1-\beta_1 x)^2} + \tfrac{1}{\delta_2}\sqrt{\tfrac{n}{4}+(\alpha_2-\beta_2 y)^2} + \right. \\ & + \tfrac{1}{\delta_1\delta_2}\cdot\tfrac{1}{(m+\beta_1)(n+\beta_2)}\sqrt{\{\tfrac{m}{4}+(\alpha_1-\beta_1 x)^2\}\{\tfrac{n}{4}+(\alpha_2-\beta_2 y)^2\}}\ \} \times \\ & \times\,\omega_{mixed}(\delta_1\delta_2),\end{aligned}$$

for any $\delta_1,\delta_2 > 0$ and any $(x,y) \in I^2$.

Proof. We apply Lemma 4 and the inequalities $x(1-x) \le \frac{1}{4}$, $y(1-y) \le \frac{1}{4}$ for any$(x,y) \in I^2$. □

REMARK 2. The inequality (9) gives us the order of the local approximation of f by $S_{m,n}f$. □

The order of the global approximation of $f \in C_b(I^2)$ by $S_{m,n}f$ is expressed in

THEOREM 4. *The GBS operator of Stancu verify the following inequality:*

$$|S_{m,n}f(x,y) - f(x,y)| \leq \tfrac{9}{4}\omega_{mixed}\left(\tfrac{\sqrt{m+4\alpha_1^2}}{m+\beta_1}, \tfrac{\sqrt{n+4\alpha_2^2}}{n+\beta_2}\right). \tag{10}$$

Proof. Taking into account that $(\alpha_1 - \beta_1 x)^2 \leq \alpha_1^2$ and $(\alpha_2 - \beta_2 y)^2 \leq \alpha_1^2$ for any $(x,y) \in I^2$, from Theorem 3, we get:

$$|S_{m,n}f(x,y) - f(x,y)| \leq$$
$$\leq \left\{ \tfrac{1}{2\delta_1}\tfrac{\sqrt{m+4\alpha_1^2}}{m+\beta_1} + \tfrac{1}{2\delta_2}\tfrac{\sqrt{n+4\alpha_2^2}}{n+\beta_2} + \tfrac{\sqrt{(m+4\alpha_1^2)(n+4\alpha_2^2)}}{4\delta_1\delta_2(m+\beta_1)(m+\beta_2)} \right\} \cdot \omega_{mixed}(\delta_1\delta_2).$$

Choosing then

$$\delta_1 = \tfrac{\sqrt{m+4\alpha_1^2}}{m+\beta_1}; \qquad \delta_2 = \tfrac{\sqrt{n+4\alpha_2^2}}{n+\beta_2};$$

it follows (10) and the proof ends. □

REMARK 3. The inequality (10) can be further refined, taking into account of the values of α_1, α_2 with respect β_1 and β_2. □

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