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THE ERROR ESTIMATION IN TERMS OF THE FIRST DERIVATIVE IN A NUMERICAL METHOD FOR THE SOLUTION OF A DELAY INTEGRAL EQUATION FROM BIOMATHEMATICS

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Abstract. The positive, bounded and smooth solution of a delay integral equation, which arise in epidemics spread, can be found in an approximative manner using numerical methods achieved by the sequence of successive approximations and quadrature rules classified after the properties of the function f, such as Lipschitzian or in the C^1 smoothness case.

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1. INTRODUCTION

The aim of this paper is to present some numerical methods for the approximation of the solution of the delay integral equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\mathrm{d}s, \quad t \in \mathbb{R}, \tag{1}$$

where $f \in C(\mathbb{R} \times \mathbb{R}_+)$.

This equation is a mathematical model for the spread of certain infectious diseases with a contact rate that varies seasonaly. Here, $x(t)$ represents the proportion of the infectivities in the population size at the time t, τ is the length of time in which an individual remains infectious and $f(t, x(t))$ is the proportion of new infectivities on unit time.

Some important results about the existence and the uniqueness of the solution of the equation (1) were given in [3], [5], [6], [7] and [8]. In [8] I.A. Rus give sufficient conditions for the existence and the uniqueness of a periodic continuous nonnegative solution. In [5] R. Precup obtains the existence of a positive solution using the Schauder's fixed point theorem, and the existence and the uniqueness of a positive solution using the Banach's fixed point theorem. In [6] and [7] R. Precup obtains existence result for a positive periodic solution using the Leray-Schauder's continuation principle, and the existence

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and uniqueness of a positive solution using the monotone iterative technique. Supposing the conditions which involve the existence of an unique positive bounded solution of the equation (1) to be satisfied, C. Iancu obtains in [4] a numerical method for the approximation of this solution using the trapezoidal quadrature formula when the function f is in the C^2 smoothness class, and give an algorithm for effective computation on the knots of this solution, together the error estimation bound.

In the Section 2 of this paper we present a theorem for the existence of an unique positive bounded solution of the equation (1). In the Section3 we obtain a trapezoid quadrature rule for functions with the first derivative Lipschitzian. Afterwards, in the Section 4, we present the error estimation in the numerical method from [4], in the case that the function f is only Lipschitzian. In the last section we present the error estimation in the assumption that $f \in C^1([-\tau, T] \times \mathbb{R}_+)$ and when $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial x}$ are Lipschitzian in each argument.

2. THE EXISTENCE AND THE UNIQUENESS OF THE POSITIVE, BOUNDED, AND SMOOTH SOLUTION

For the equation (1) we consider the initial condition $x(t) = \Phi(t)$, $\forall t \in [-\tau, 0]$, which means that the proportion $\Phi(t)$ of infectives in population is known for $t \in [-\tau, 0]$. Let $T > 0$ be a fixed real number. We have the initial value problem:

$$\begin{cases} x(t) = \int_{t-\tau}^{t} f(s, x(s)) \mathrm{d}s, \ \forall t \in [0, T] \\ x(t) = \Phi(t), \ \forall t \in [-\tau, 0], \end{cases} \tag{2}$$

where $\Phi \in C[-\tau, 0]$ and satisfy the condition:

$$\Phi(0) = b = \int_{-\tau}^{0} f(s, \Phi(s)) \mathrm{d}s. \tag{3}$$

Since $f \in C\left([-\tau, T] \times \mathbb{R}\right)$, using the Leibnitz's formula of derivation into an integral with parameters, we can see that the initial value problem (2) leads to a Cauchy's problem for a delay differential equation:

$$\begin{cases} x'(t) = f(t, x(t)) - f(t-\tau, x(t-\tau)), \ \forall t \in [0, T] \\ x(t) = \Phi(t), \ \forall t \in [-\tau, 0] \end{cases} \tag{4}$$

and (2) and (4) are equivalent.

To obtain the existence of an unique positive, bounded solution we list these assumptions:

(i) $\exists M > 0, \exists \beta \geq a > 0$ such that Φ is continuous on $[-\tau, 0]$ and $0 < a \leq \Phi(t) \leq \beta$, $\forall t \in [-\tau, 0]$;

(ii) f have the properties:

$$f \in C\left([-\tau, T] \times [a, \beta]\right), \ f(t, x) \geq 0, \quad \forall t \in [-\tau, T], \quad \forall x \geq 0,$$
$$f(t, u) \leq M, \quad \forall t \in [-\tau, T], \quad \forall u \in [a, \beta];$$

(iii) Φ satisfies the condition (3), $\Phi \in C^1[-\tau, 0]$ and
$$\Phi'(0) = f(0, b) - f(-\tau, \Phi(-\tau));$$

(iv) $M\tau \leq \beta$ and there exists an integrable function $g(t)$ such that
$$f(t, x) \geq g(t), \quad \forall t \in [-\tau, T], \quad \forall x \geq a$$
with
$$\int_{t-\tau}^{t} g(s)\mathrm{d}s \geq a, \ \ \forall t \in [0, T];$$

(v) $\exists L_1 > 0$ such that $|f(t, x) - f(t, y)| \leq L_1 |x - y|$ for all $t \in [-\tau, T]$ and $x, y \in [a, +\infty)$.

Theorem 1. *If the condition* (3) *and the above conditions* (i)–(v) *are satisfied, then the equation* (1) *has a unique continuous on* $[-\tau, T]$ *solution* $x(t)$*, with* $a \leq x(t) \leq \beta$*,* $\forall t \in [-\tau, T]$*, such that* $x(t) = \Phi(t)$ *for* $t \in [-\tau, 0]$*. Moreover,*
$$\max\{|x_n(t) - x(t)| : t \in [0, T]\} \longrightarrow 0$$
as $n \to \infty$*, where* $x_n(t) = \Phi(t)$ *for* $t \in [-\tau, 0]$*,* $n \in \mathbb{N}$*,* $x_0(t) = b$ *and* $x_n(t) = \int_{t-\tau}^{t} f(s, x_{n-1}(s))\mathrm{d}s$ *for* $t \in [0, T]$*,* $n \in \mathbb{N}^*$*. The solution* x *belongs to* $C^1[-\tau, T]$*.*

Proof. As in [5] under the conditions (i), (ii), (iv), (v) follow the existence of an unique positive continuous on $[-\tau, T]$ solution for (1) such that $x(t) \geq a, \forall t \in [-\tau, T]$ and $x(t) = \Phi(t)$ for $t \in [-\tau, 0]$. Using Theorem 2 from [5] we infer that
$$\max\{|x_n(t) - x(t)| : t \in [0, T]\} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$
From (iv) we see that
$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\mathrm{d}s \leq \int_{t-\tau}^{t} M\mathrm{d}s = M\tau \leq \beta, \quad \forall t \in [0, T].$$

Because $a \leq \Phi(t) \leq \beta, \forall t \in [-\tau, 0]$ and $x(t) = \Phi(t)$ for $t \in [-\tau, 0]$ we infer that $a \leq x(t) \leq \beta, \forall t \in [-\tau, T]$, and the solution is bounded. We know that x is solution for (1) and we have $x(t) = \int_{t-\tau}^{t} f(s, x(s))\mathrm{d}s$, for all $t \in [0, T]$, and because $f \in C([-\tau, T] \times [a, \beta])$ we infer that x is derivable on $[0, T]$, and x' is continuous on $[0, T]$. From the condition (iii) follow that x is derivable with x' continuous on $[-\tau, 0]$ (including the continuity in the point $t = 0$). Then $x \in C^1[-\tau, T]$ and this concludes the proof. □

Remark 1. In the hypotheses of Theorem 1 the initial value problem (2) has an unique positive, bounded and smooth solution on the interval $[-\tau, T]$. As in [8], this solution depends continuously by f. □

Consider the functions $F_m : [-\tau, T] \to \mathbb{R}, F_m(t) = f(t, x_m(t))$, $\forall m \in \mathbb{N}$. Suppose that $\exists L_2 > 0$ such that

$$|f(t_1, x) - f(t_2, x)| \leq L_2 |t_1 - t_2|, \quad \forall t_1, t_2 \in [-\tau, T], \forall x \in [a, \beta]. \tag{5}$$

Proposition 2. *If the conditions* (i)–(v) *and* (5) *holds, then the all functions* $F_m, m \in \mathbb{N}$, *are Lipschitzian with the Lipschitz constant* $L = L_2 + 2M_0L_1$.

Proof. From the condition (v) we have,

$$|f(t, x_m(t)) - f(t, y_m(t))| \leq L_1 |x_m(t) - y_m(t)|, \quad \forall t \in [-\tau, T].$$

Let

$$M_0 = \max\{\|\Phi'\|, M\}.$$

We can see that

$$\begin{aligned}
|x_m(t_1) - x_m(t_2)| &= \left| \int_{t_1-\tau}^{t_1} f(s, x_{m-1}(s))\mathrm{d}s - \int_{t_2-\tau}^{t_2} f(s, x_{m-1}(s))\mathrm{d}s \right| \\
&= \left| \int_{t_1-\tau}^{t_2-\tau} f(s, x_{m-1}(s))\mathrm{d}s + \int_{t_2-\tau}^{t_1} f(s, x_{m-1}(s))\mathrm{d}s \right. \\
&\quad \left. - \int_{t_2-\tau}^{t_1} f(s, x_{m-1}(s))\mathrm{d}s - \int_{t_1}^{t_2} f(s, x_{m-1}(s))\mathrm{d}s \right| \\
&\leq \int_{t_1-\tau}^{t_2-\tau} |f(s, x_{m-1}(s))| \,\mathrm{d}s + \int_{t_1}^{t_2} |f(s, x_{m-1}(s))| \,\mathrm{d}s \\
&\leq 2M_0 \cdot |t_1 - t_2|, \qquad \forall t_1, t_2 \in [-\tau, T].
\end{aligned}$$

Then

$$\begin{aligned}
|F_m(t_1) - F_m(t_2)| &= |f(t_1, x_m(t_1)) - f(t_2, x_m(t_2))| \\
&\leq |f(t_1, x_m(t_1)) - f(t_2, x_m(t_1))| \\
&\quad + |f(t_2, x_m(t_1)) - f(t_2, x_m(t_2))| \\
&\leq (L_2 + 2M_0L_1) |t_1 - t_2|, \quad \forall t_1, t_2 \in [-\tau, T],
\end{aligned}$$

and so F_m is Lipschitzian, $\forall m \in \mathbb{N}$. □

3. A TRAPEZOID QUADRATURE RULE FOR FUNCTIONS WITH THE FIRST DERIVATIVE LIPSCHITZIAN

For Lipschitzian functions and for smooth functions (in the class C^1) the trapezoid quadrature rules are presented in [1] and [2]. Here we obtain a new quadrature rule for functions $f \notin C^2[a, b]$, but with f' Lipschitzian.

Lemma 3. *For a derivable function* $f : [a, b] \to \mathbb{R}$ *having Lipschitzian derivative with a Lipschitz's constant* $L \geq 0$, *the following quadrature formula holds:*

$$\int_a^b f(t)\mathrm{d}t = \tfrac{b-a}{2}[f(a) + f(b)] + R(f),$$

and

$$|R(f)| \leq \tfrac{L(b-a)^3}{12}.$$

Proof. We compute by parts the following integral,

$$\int_a^b (t - \tfrac{a+b}{2})[f'(t) - f'(\tfrac{a+b}{2})]\mathrm{d}t = \int_a^b (t - \tfrac{a+b}{2}) f'(t)\mathrm{d}t$$
$$= f(t) \cdot (t - \tfrac{a+b}{2}) \mid_a^b - \int_a^b f(t)\mathrm{d}t$$
$$= \tfrac{(b-a)}{2}[f(a) + f(b)] - \int_a^b f(t)\mathrm{d}t,$$

since

$$\int_a^b (t - \tfrac{a+b}{2}) f'(\tfrac{a+b}{2})\mathrm{d}t = f'(\tfrac{a+b}{2}) \int_a^b (t - \tfrac{a+b}{2})\mathrm{d}t = 0.$$

Then, we have

$$\int_a^b f(t)\mathrm{d}t = \tfrac{(b-a)}{2}[f(a) + f(b)] - \int_a^b (t - \tfrac{a+b}{2})[f'(t) - f'(\tfrac{a+b}{2})]\mathrm{d}t$$
$$= \tfrac{(b-a)}{2}[f(a) + f(b)] + R(f),$$

with

$$R(f) = -\int_a^b (t - \tfrac{a+b}{2})[f'(t) - f'(\tfrac{a+b}{2})]\mathrm{d}t.$$

For the remainder estimation we have,

$$\mid R(f) \mid = \left| \int_a^b (t - \tfrac{a+b}{2})[f'(t) - f'(\tfrac{a+b}{2})]\mathrm{d}t \right|$$
$$\leq \int_a^b \mid t - \tfrac{a+b}{2} \mid \cdot \mid f'(t) - f'(\tfrac{a+b}{2}) \mid \mathrm{d}t$$
$$\leq L \int_a^b \mid t - \tfrac{a+b}{2} \mid^2 \mathrm{d}t$$
$$= L\Big[\int_a^{\frac{a+b}{2}} (\tfrac{a+b}{2} - t)^2\mathrm{d}t + \int_{\frac{a+b}{2}}^b (t - \tfrac{a+b}{2})^2\mathrm{d}t\Big]$$
$$= \tfrac{L(b-a)^3}{12}.$$

□

Corollary 4. *In the conditions of the previous lemma, considering a partition of the interval* $[a, b]$,

$$a = t_0 < t_1 < ... < t_{n-1} < t_n = b,$$

we have the quadrature rule

$$\int_a^b f(t)\mathrm{d}t = \tfrac{1}{2} \sum_{i=0}^{n-1} (t_{i+1} - t_i)[f(t_i) - f(t_{i+1})] + R_n(f),$$

with the remainder estimation,

$$\mid R_n(f) \mid \leq \tfrac{L}{12} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^3.$$

Proof. On the each interval $[t_i, t_{i+1}]$, $i = \overline{0, n-1}$, we have

$$\int_{t_i}^{t_{i+1}} f(t)\mathrm{d}t = \tfrac{1}{2} \cdot (t_{i+1} - t_i)[f(t_i) - f(t_{i+1})] + R_i(f).$$

Then

$$\int_a^b f(t)\mathrm{d}t = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t)\mathrm{d}t = \tfrac{1}{2} \sum_{i=0}^{n-1} (t_{i+1} - t_i)[f(t_i) - f(t_{i+1})] + \sum_{i=0}^{n-1} R_i(f)$$

and

$$\mid R_n(f) \mid \leq \sum_{i=0}^{n-1} \mid R_i(f) \mid \leq \tfrac{L}{12} \sum_{i=0}^{n-1} (t_{i+1} - t_i)^3.$$

□

REMARK 2. If the partition of $[a, b]$ is uniform, then $t_i = a + i \cdot \frac{b-a}{n}$, $\forall i = \overline{0, n}$ and $t_{i+1} - t_i = \frac{b-a}{n}$, $\forall i = \overline{0, n-1}$. In this case, the quadrature rule is:

$$\int_a^b f(t)\mathrm{d}t = \tfrac{b-a}{2n} \left[f(a) + 2 \sum_{i=1}^{n-1} f(t_i) + f(b) \right] + R_n(f), \tag{6}$$

with the remainder estimation:

$$\mid R_n(f) \mid \leq \tfrac{L(b-a)^3}{12n^2}. \tag{7}$$

□

4. THE ERROR ESTIMATION IN THE LIPSCHITZIAN CASE

Let φ be the solution of (2), which, by virtue of Theorem 1, can be obtained by the successive approximations method.

So, we have $\varphi(t) = \varphi_m(t) = \Phi(t)$, $\forall m \in \mathbb{N}^*$, for $t \in [-\tau, 0]$ and,

$$\begin{cases} \varphi_0(t) = \Phi(0) = b = \int_{-\tau}^{0} f(s, \Phi(s))\mathrm{d}s \\ \varphi_1(t) = \int_{t-\tau}^{t} f(s, \varphi_0(s))\mathrm{d}s \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varphi_m(t) = \int_{t-\tau}^{t} f(s, \varphi_{m-1}(s))\mathrm{d}s \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \quad \forall m \in \mathbb{N}^*, \quad \forall t \in [0, T]. \tag{8}$$

To obtain the sequence of successive approximations (8), it is necessary to calculate the integrals which appear in the right-hand side, but this problem can be solved only approximatively. In this purpose can be use quadrature formulae rules. This rules will be adapted in correlation with the properties of the function f.

First, we consider functions f Lipschitzian, with a constant $L > 0$. We can apply the following quadrature formula:

$$\int_a^b f(x)\mathrm{d}x = \tfrac{b-a}{2} \left[f(a) + f(b) \right] + R(f)$$

with the remainder estimation

$$|R(f)| \leq \tfrac{(b-a)^2}{4} \cdot L,$$

and the generalized quadrature rule obtained using an uniform partition of the interval $[a,b]$ (see [1] and [2]).

Considering an equidistant division, $a = x_0 < x_1 < ... < x_n = b$, $n \in \mathbb{N}^*$, the quadrature rule becomes

$$\int_a^b f(x)\mathrm{d}x = \tfrac{(b-a)}{2n}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right] + R_n(f), \tag{9}$$

where the remainder $R_n(f)$ satisfies the inequality:

$$|R_n(f)| \le \tfrac{(b-a)^2 \cdot L}{4n}.$$

To compute the integrals from (8) we consider instead the function f, the functions F_m, which by virtue of Proposition 2 are Lipschitzian.

We consider an uniform partition of the interval $[-\tau, 0]$ by the points:

$$-\tau = t_0 < t_1 < ... < t_{n-1} < t_n = 0,$$

where $t_i - t_{i-1} = h$, $h = \frac{\tau}{n}$, $\forall i = \overline{1,n}$, $n \in \mathbb{N}^*$. Supposing that $T = l\tau$, $l \in \mathbb{N}^*$ we construct a uniform partition of $[0,T]$, with $q = n + l \cdot n$,

$$0 = t_n < t_{n+1} < ... < t_{q-1} < t_q = T,$$

with $t_i - t_{i-1} = h = \frac{\tau}{n}$, $\forall i = \overline{n+1,q}$. We can see that $t_k - \tau = t_{k-n}$, $\forall k = \overline{n,q}$.

Using the quadrature rule (9) to compute $\varphi_m(t)$ from (8) for $t = t_k$, $k = \overline{n+1,q}$, we have:

$$\begin{aligned}\varphi_m(t_k) =& \int_{t_k-\tau}^{t_k} f(s, \varphi_{m-1}(s))\mathrm{d}s \\ =& \tfrac{\tau}{2n} \cdot \Bigg[f(t_k - \tau, \varphi_{m-1}(t_k - \tau)) + f(t_k, \varphi_{m-1}(t_k)) \\ &+ 2\sum_{i=1}^{n-1} f(t_{k+i} - \tau, \varphi_{m-1}(t_{k+i} - \tau))\Bigg] + r_{m,k}(f), \quad \forall m \in \mathbb{N}^*,\end{aligned}$$

where for the remainder $r_{m,k}(f)$ we have the estimation

$$|r_{m,k}(f)| \le \tfrac{\tau^2 L}{4n}, \quad \forall k = \overline{n+1,q}, \ \forall m \in \mathbb{N}^*. \tag{10}$$

In [4] C. Iancu presents the following algorithm:

$$\begin{aligned}\varphi_1(t_k) =& \tfrac{\tau}{2n}[f\,(t_k - \tau, \varphi_0(t_k - \tau)) + f\,(t_k, \varphi_0(t_k)) \\ &+ 2\sum_{i=1}^{n-1} f(t_{k+i} - \tau, , \varphi_0(t_{k+i} - \tau))] + r_{1,k}(f) \\ =& \tfrac{\tau}{2n}[f\,(t_{k-n}, \varphi_0(t_{k-n})) + f\,(t_k, \varphi_0(t_k)) + \\ &+ 2\sum_{i=1}^{n-1} f(t_{k+i-n}, \varphi_0(t_{k+i-n}))] + r_{1,k}(f) \\ =& \widetilde{\varphi_1}(t_k) + r_{1,k}(f), \quad \forall k = \overline{n+1,q},\end{aligned}$$

and

$$\begin{aligned}\varphi_2(t_k) = {} & \tfrac{\tau}{2n}[f\left(t_k-\tau,\widetilde{\varphi_1}(t_k-\tau)+r_{1,k-n}(f)\right)+f(t_k,\widetilde{\varphi_1}(t_k)+r_{1,k}(f)) \\ & +2\sum_{i=1}^{n-1} f\left(t_{k+i}-\tau,\widetilde{\varphi_1}(t_{k+i}-\tau)+r_{1,k+i-n}(f)\right)]+r_{2,k}(f) \\ & \tfrac{\tau}{2n}[f\left(t_{k-n},\widetilde{\varphi_1}(t_{k-n})\right)+f\left(t_k,\widetilde{\varphi_1}(t_k)\right) \\ & = 2\sum_{i=1}^{n-1} f(t_{k+i-n},,\widetilde{\varphi_1}(t_{k+i-n}))]+\widetilde{r_{2,k}}(f) \\ & = \widetilde{\varphi_2}(t_k)+\widetilde{r_{2,k}}(f), \quad \forall k=\overline{n+1,q}.\end{aligned}$$

By induction, for $m \geq 3$, it follows:

$$\begin{aligned}\varphi_m(t_k) = {} & \tfrac{\tau}{2n}[f\left(t_k-\tau,\widetilde{\varphi_{m-1}}(t_k-\tau)\right)+f\left(t_k,\widetilde{\varphi_{m-1}}(t_k)\right) \\ & +2\sum_{i=1}^{n-1} f t_{k+i}-\tau,,\widetilde{\varphi_{m-1}}(t_{k+i}-\tau))]+\widetilde{r_{m,k}}(f) \\ = {} & \tfrac{\tau}{2n}[f\left(t_{k-n},\widetilde{\varphi_{m-1}}(t_{k-n})\right)+f\left(t_k,\widetilde{\varphi_{m-1}}(t_k)\right) \\ & +2\sum_{i=1}^{n-1} f t_{k+i-n},\widetilde{\varphi_{m-1}}(t_{k+i-n}))]+\widetilde{r_{m,k}}(f) \\ = {} & \widetilde{\varphi_m}(t_k)+\widetilde{r_{m,k}}(f), \quad \forall k=\overline{n+1,q}. \end{aligned} \tag{11}$$

To estimate the remainder, he use the classic quadrature rule (for functions $f \in C^2$). For the error estimation of the remainders, $\widetilde{r_{m,k}}(f)$, we obtain in this case the estimations:

$$\begin{aligned}\mid \widetilde{r_{2,k}}(f) \mid \leq {} & \tfrac{\tau}{2n}[L_1\cdot \mid r_{1,k-n}(f) \mid +2\sum_{i=1}^{n-1} L_1\cdot \mid r_{1,k+i-n}(f) \mid + \\ & + L_1\cdot \mid r_{1,k}(f) \mid]+ \mid r_{2,k}(f) \mid \\ & \leq \tfrac{\tau L_1}{2n}[\tfrac{\tau^2 L}{4n}+2(n-1)\tfrac{\tau^2 L}{4n}+\tfrac{\tau^2 L}{4n}]+\tfrac{\tau^2 L}{4n}\end{aligned}$$

$$= \frac{\tau^2 L}{4n}(\tau L_1 + 1), \quad \forall k = \overline{n+1, q},$$

and for $m \geq 3$, by induction, we obtain:

$$|\widetilde{r_{m,k}}(f)| = |\varphi_m(t_k) - \widetilde{\varphi_m}(t_k)| \leq \frac{\tau^2 L}{4n}(\tau^{m-1} \cdot L_1^{m-1} + ... + \tau L_1 + 1),$$

$\forall k = \overline{n+1, q}$, $\forall m \in \mathbb{N}^*$.

If $\tau L_1 < 1$, we have

$$|\widetilde{r_{m,k}}(f)| \leq \frac{\tau^2 L}{4n} \cdot \left(\frac{1-\tau^m L_1^m}{1-\tau L_1}\right) \leq \frac{\tau^2 L}{4n(1-\tau L_1)} \tag{12}$$

and we get the sequence $(\widetilde{\varphi_m}(t_k))_{m\in\mathbb{N}^*}$, $k = \overline{n+1, q}$ which approximates the sequence of successive approximations (8) on the knots $t_k, k = \overline{n+1, q}$, with the error

$$|\varphi_m(t_k) - \widetilde{\varphi_m}(t_k)| \leq \frac{\tau^2 L}{4n(1-\tau L_1)}. \tag{13}$$

THEOREM 5. *Consider the initial value problem* (2) *under the conditions* (i)–(v)*,* (3)*,* (5) *and* $\tau L_1 < 1$*. If the exact solution* φ *is approximated by the sequence* $(\widetilde{\varphi_m}(t_k))_{m\in\mathbb{N}^*}$*,* $k = \overline{n+1, q}$*, on the knots* $t_k, k = \overline{n+1, q}$*, by the successive approximations method* (8) *combined with the quadrature rule* (9)*, then the error estimation is:*

$$|\varphi_m(t_k) - \widetilde{\varphi_m}(t_k)| \leq \frac{\tau^2}{1-\tau L_1}\left[\frac{L}{4n} + \tau^{m-2} \cdot L_1^m \cdot \|\varphi_0 - \varphi_1\|_{C[0,T]}\right], \tag{14}$$

$\forall m \in \mathbb{N}^*$*,* $k = \overline{n+1, q}$*.*

Proof. By the Banach's principle and Picard's theorem (Theorem 1.4.4, page 33, from [9]) we have the following estimation:

$$|\varphi(t_k) - \varphi_m(t_k)| \leq \frac{\tau^m L_1^m}{1-\tau L_1} \cdot \|\varphi_0 - \varphi_1\|_{C[0,T]}.$$

Then, by virtue of the (13) estimation we have:

$$\begin{aligned}|\varphi(t_k) - \widetilde{\varphi_m}(t_k)| &\leq |\varphi(t_k) - \varphi_m(t_k)| + |\varphi_m(t_k) - \widetilde{\varphi_m}(t_k)| \\ &\leq \frac{\tau^m L_1^m}{1-\tau L_1} \cdot \|\varphi_0 - \varphi_1\|_{C[0,T]} + \frac{\tau^2 L}{4n(1-\tau L_1)} \\ &= \frac{\tau^2}{1-\tau L_1}\left[\frac{L}{4n} + \tau^{m-2} \cdot L_1^m \cdot \|\varphi_0 - \varphi_1\|_{C[0,T]}\right].\end{aligned}$$

□

We can approximate also, the values of the derivative of the solution on the knots. Indeed, from (8) we have

$$\varphi_m(t) = \int_{t-\tau}^{t} f(s, \varphi_{m-1}(s))\mathrm{d}s, \quad \forall t \in [0, T], \quad \forall m \in \mathbb{N}^*.$$

By differentiation we obtain:

$$\varphi_m'(t) = f(t, \varphi_{m-1}(t)) - f(t-\tau, \varphi_{m-1}(t-\tau)), \quad \forall t \in [0, T], \quad \forall m \in \mathbb{N}^*. \tag{15}$$

For $t = t_k, k = \overline{n+1, q}$, we have

$$\varphi'_m(t_k) = f(t_k, \widetilde{\varphi_{m-1}}(t_k) + \widetilde{r_{m-1,k}}(f)) - f(t_k - \tau, \widetilde{\varphi_{m-1}}(t_k - \tau) + \widetilde{r_{m-1,k-n}}(f))$$
$$= f(t_k, \widetilde{\varphi_{m-1}}(t_k)) - f(t_{k-n}, \widetilde{\varphi_{m-1}}(t_{k-n})) + \varpi_{m,k}(f)$$

(16)

$$= \widetilde{\varphi'_m}(t_k) + \varpi_{m,k}(f), \quad \forall m \in \mathbb{N}^*, \quad k = \overline{n+1, q}.$$

For the remainder $\varpi_{m,k}(f)$ we have the estimation

$$(17) \qquad |\ \varpi_{m,k}(f)\ | \leq 2L_1 \cdot \left|\widetilde{r_{m-1,k}}(f)\right| \leq \tfrac{2L_1 \cdot \tau^2 L}{4n(1-\tau L_1)}, \quad \forall m \in \mathbb{N}^*,\ k = \overline{n+1, q}.$$

Since $\varphi'(t_k) = f(t_k, \varphi(t_k)) - f(t_k - \tau, \varphi(t_k - \tau)), \quad \forall k = \overline{n+1, q}$, by (15) we get

$$|\ \varphi'(t_k) - \varphi'_m(t_k)\ | \leq L_1(|\varphi(t_k) - \varphi_{m-1}(t_k)| + |\varphi(t_k - \tau) - \varphi_{m-1}(t_k - \tau)|),$$

$\forall m \in \mathbb{N}^*,\ k = \overline{n+1, q}$. We can see that

$$|\varphi(t_k) - \varphi_{m-1}(t_k)| \leq \tfrac{\tau^m L_1^m}{1-\tau L_1} \cdot \|\varphi_0 - \varphi_1\|_{C[0,T]}$$

and

$$|\varphi(t_k - \tau) - \varphi_{m-1}(t_k - \tau)| \leq \tfrac{\tau^m L_1^m}{1-\tau L_1} \cdot \|\varphi_0 - \varphi_1\|_{C[0,T]},$$

$\forall m \in \mathbb{N}^*, k = \overline{n+1, q}$, where

$$\|\varphi_0 - \varphi_1\|_{C[0,T]} = \max_{t \in [0,T]} \left| \int_{t-\tau}^{t} f(s, b)\mathrm{d}s - b \right| \leq b + M\tau.$$

For this reason,

$$|\ \varphi'(t_k) - \widetilde{\varphi'_m}(t_k)\ | \leq |\ \varphi'(t_k) - \varphi'_m(t_k)\ | + |\ \varphi'_m(t_k) - \widetilde{\varphi'_m}(t_k)\ |$$

$$(18) \qquad \leq 2L_1 \cdot \tfrac{\tau^m L_1^m (b+M\tau)}{1-\tau L_1} + \tfrac{2L_1 \cdot \tau^2 L}{4n(1-\tau L_1)}, \quad \forall m \in \mathbb{N}^*,\ k = \overline{n+1, q}.$$

REMARK 3. From these above, we see that it is possible to compute the values $\widetilde{\varphi_m}(t_k)$ given in (11) and $\widetilde{\varphi'_m}(t_k)$ given in (16) on the knots $t_k, k = \overline{n+1, q}$, approximating the solution and his derivative with the error estimations as in (13) and (18), respectively. □

5. THE ERROR ESTIMATION USING THE FIRST DERIVATIVE

Suppose that $f \in C^1([-\tau, T] \times [a, \beta])$, and consequently there exists $M_{10} \geq 0, M_{01} \geq 0$ such that

$$M_{10} = \max\{\left|\tfrac{\partial f}{\partial t}(t, x(t))\right| : t \in [-\tau, T]\},$$
$$M_{01} = \max\{\left|\tfrac{\partial f}{\partial x}(t, x(t))\right| : t \in [-\tau, T]\}.$$

Since

$$[f(t, x(t))]' = \frac{\partial f}{\partial t}(t, x(t)) + \frac{\partial f}{\partial x}(t, x(t)) \cdot x'(t)$$

and

$$x'(t) = f(t, x(t)) - f(t - \tau, x(t - \tau)),$$

then

$$| F'_m(t) |= \left|[f(t, x_m(t))]'\right| \leq M_{10} + 2M_{01} \cdot M, \ \ \forall t \in [-\tau, T], \ \forall m \in \mathbb{N}.$$

Therefore

$$\| F'_m \| = \max\{| F'_m(t) |: t \in [-\tau, T]\} \leq M_{10} + 2M_{01} \cdot M, \ \ \forall m \in \mathbb{N}. \tag{19}$$

We can use the following trapezoid quadrature formula (as in [1] and [2]), when $f \in C^1[a, b]$:

$$\int_a^b f(x)\mathrm{d}x = \tfrac{b-a}{2n}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(a + \tfrac{i(b-a)}{n})\right] + R_n(f) \tag{20}$$

with the error estimation

$$|R_n(f)| \leq \tfrac{(b-a)^2}{4n} \cdot \|f'\| \,. \tag{21}$$

Considering the algorithm (11) adapted at quadrature rule (20) with the remainder estimation (21) we obtain the following result:

Theorem 6. *Consider the initial value problem* (2) *under the conditions* (i)–(v) *and* $f \in C^1([-\tau, T] \times [a, \beta])$. *If the exact solution* φ *is approximated by the sequence* $(\widetilde{\varphi_m}(t_k))_{m\in\mathbb{N}}$, $k = \overline{n+1, q}$, *on the knots* $t_k, k = \overline{n+1, q}$, *by the successive approximations method* (8) *combined with the quadrature rule* (20) *and the remainder estimation* (21), *then the error estimation is:*

$$|\varphi(t_k) - \widetilde{\varphi_m}(t_k)| \leq \tfrac{\tau^2}{1-\tau L_1} \cdot [\tfrac{M_{10}+2M\cdot M_{01}}{4n} + \tau^{m-2} \cdot L_1^m \cdot \|\varphi_0 - \varphi_1\|],$$

$\forall k = \overline{n+1, q}, \ m \in \mathbb{N}^*$.

Proof. We have, analogous with the proof of the Theorem 5, and according to the above quadrature rule, the following:

$$\begin{aligned}|\varphi(t_k) - \widetilde{\varphi_m}(t_k)| &\leq |\varphi(t_k) - \varphi_m(t_k)| + |\varphi_m(t_k) - \widetilde{\varphi_m}(t_k)| \\ &\leq \tfrac{\tau^m L_1^m}{1-\tau L_1} \cdot \|\varphi_0 - \varphi_1\| + \tfrac{\tau^2(M_{10}+2M\cdot M_{01})}{4n(1-\tau L_1)} \\ &= \tfrac{\tau^2}{1-\tau L_1}[\tfrac{M_{10}+2M\cdot M_{01}}{4n} + \tau^{m-2} \cdot L_1^m \cdot \|\varphi_0 - \varphi_1\|]. \qquad \square\end{aligned}$$

In the case $f \in C^2([-\tau, T] \times [a, \beta])$, using the classic trapezoidal formula, C. Iancu obtains in [4] the result:

Theorem 7 ([4, Th. 5.1]). *Consider the initial value problem* (2) *under the conditions* (i)–(v) *and* $f \in C^2([-\tau, T] \times [a, \beta])$. *If the exact solution* φ *is approximated by the sequence* $(\widetilde{\varphi_m}(t_k))_{m\in\mathbb{N}}$, $k = \overline{n+1, q}$, *on the knots* $t_k, k = \overline{n+1, q}$, *by the successive approximations method* (8), *combined with* *the trapezoidal rule which give on the knots the values,*

$$\varphi_m(t_k) = \frac{\tau}{2n}[f\left(t_k - \tau, \widetilde{\varphi_{m-1}}(t_k - \tau)\right) + f\left(t_k, \widetilde{\varphi_{m-1}}(t_k)\right)$$
$$+ 2\sum_{i=1}^{n-1} f(t_i, , \widetilde{\varphi_{m-1}}(t_i)] + \widetilde{r_{m,k}}(f)$$
$$= \widetilde{\varphi_m}(t_k) + \widetilde{r_{m,k}}(f), \quad \forall k = \overline{n+1, q}, m \in \mathbb{N}^*,$$

then the following error estimation holds:

$$|\varphi(t_k) - \widetilde{\varphi_m}(t_k)| \leq \frac{\tau^3}{1-\tau L_1}\left[\tau^{m-3} \cdot L_1^m \cdot \|\varphi_0 - \varphi_1\|_{C[0,T]} + \frac{M_*}{12n^2}\right],$$

$\forall k = \overline{n+1, q}$, $m \in \mathbb{N}^*$, *where* $M_* = M_1 + 3\tau M_1^2 + \tau^2 M_1^3$ *and*

$$M_1 = \max\left\{\left|\frac{\partial^\alpha f(s,u)}{\partial s^{\alpha_1}\partial u^{\alpha_2}}\right|; |\alpha| \leq 2, s \in [0,T], u \in [a,\beta]\right\}.$$

We apply now, the quadrature rule (6)–(7) to compute the integrals from the successive approximations (8), in the case that F'_m are Lipschitzian, using the algorithm (11).

Proposition 8. *Suppose that the following conditions holds:*

a) $f \in C^1([-\tau, T] \times [a, \beta])$, $\Phi \in C^2[-\tau, 0]$;
b) $\exists L_{11} \geq 0, L_{10} \geq 0$ *such that*

$$\left|\frac{\partial f}{\partial t}(t_1, x) - \frac{\partial f}{\partial t}(t_2, x)\right| \leq L_{11}|t_1 - t_2|, \quad \forall t_1, t_2 \in [-\tau, T], \forall x \in [a, \beta],$$
$$\left|\frac{\partial f}{\partial t}(t, x_1) - \frac{\partial f}{\partial t}(t, x_2)\right| \leq L_{10}|x_1 - x_2|, \quad \forall t \in [-\tau, T], \forall x_1 x_2 \in [a, \beta];$$

c) $\exists L_{21} \geq 0, L_{20} \geq 0$ *such that*

$$\left|\frac{\partial f}{\partial x}(t_1, x) - \frac{\partial f}{\partial x}(t_2, x)\right| \leq L_{21}|t_1 - t_2|, \quad \forall t_1, t_2 \in [-\tau, T], \forall x \in [a, \beta],$$
$$\left|\frac{\partial f}{\partial x}(t, x_1) - \frac{\partial f}{\partial x}(t, x_2)\right| \leq L_{20}|x_1 - x_2|, \quad \forall t \in [-\tau, T], \forall x_1 x_2 \in [a, \beta].$$

Then the functions F'_m *are Lipschitzian on* $[-\tau, T]$.

Proof. We have,

$$\left|F'_m(t_1) - F'_m(t_2)\right| =$$
$$= \left| \frac{\partial f}{\partial t}(t_1, x_m(t_1)) + \frac{\partial f}{\partial x}(t_1, x_m(t_1)) \cdot x'_m(t_1) - \right.$$
$$\left. - \frac{\partial f}{\partial t}(t_2, x_m(t_2)) - \frac{\partial f}{\partial x}(t_1, x_m(t_2)) \cdot x'_m(t_2) \right|$$
$$\leq L_{11}|t_1 - t_2| + L_{10}|x(t_1) - x(t_2)| + \| \frac{\partial f}{\partial x} \| \cdot |x'_m(t_1) - x'_m(t_2)|$$
$$+ |x'_m(t_2)| \cdot \left[\left|\frac{\partial f}{\partial x}(t_1, x_m(t_1)) - \frac{\partial f}{\partial x}(t_2, x_m(t_1))\right|\right.$$

$$+\left|\frac{\partial f}{\partial x}(t_2,x_m(t_1))-\frac{\partial f}{\partial x}(t_2,x_m(t_2))\right|\Big]$$
$$\leq L_{11}\left|t_1-t_2\right|+2M\cdot L_{10}\left|t_1-t_2\right|+\parallel \frac{\partial f}{\partial x}\parallel\cdot(2L_2+4L_1M)\cdot\left|t_1-t_2\right|$$
$$+\parallel x_m'\parallel\left(L_{21}\left|t_1-t_2\right|+L_{20}\left|x_m(t_1)-x_m(t_2)\right|\right)$$
$$\leq\left[L_{11}+2M\cdot L_{10}+2M(L_{21}+2ML_{20})+M_{01}(2L_2+4L_1M)\right]\cdot\left|t_1-t_2\right|,$$

$\forall t_1,t_2\in[-\tau,T]$, that is,

$$\left|F_m'(t_1)-F_m'(t_2)\right|\leq L'\left|t_1-t_2\right|,\quad \forall t_1,t_2\in[-\tau,T].$$

On the other hand, since$f\in C^1([-\tau,T]\times[a,\beta])$, we have

$$x'(t)=f(t,x(t))-f(t-\tau,x(t-\tau)),\quad \forall t\in[-\tau,T].$$

$\parallel x_m'\parallel\leq 2M$ and

$$\left|x_m'(t_1)-x_m'(t_2)\right|\leq\left|f(t_1,x_{m-1}(t_1))-f(t_2,x_{m-1}(t_1))\right|$$
$$+\left|f(t_2,x_{m-1}(t_1))-f(t_2,x_{m-1}(t_2))\right|$$
$$+\mid f(t_1-\tau,x_{m-1}(t_1-\tau))-f(t_2-\tau,x_{m-1}(t_1-\tau))\mid$$
$$+\mid f(t_2-\tau,x_{m-1}(t_1-\tau))-f(t_2-\tau,x_{m-1}(t_2-\tau))\mid$$
$$\leq(2L_2+4L_1M)\cdot\left|t_1-t_2\right|,\ \ \forall t_1,t_2\in[-\tau,T],\quad \forall m\in\mathbb{N}^*.$$

The Lipschitz constant is

$$L'=\left[L_{11}+2M\cdot L_{10}+2M(L_{21}+2ML_{20})+M_{01}(2L_2+4L_1M)\right].\qquad\square$$

THEOREM 9. *In the conditions of Theorem* 5 *and Proposition* 8*, the exact solution of the initial value problem* (2)*,* φ*, is approximated on the knots* $t_k, k=\overline{n+1,q}$*, by the sequence* $\left(\widetilde{\varphi_m}(t_k)\right), m\in\mathbb{N}$*, given by* (11)*, with the error:*

$$\left|\varphi(t_k)-\widetilde{\varphi_m}(t_k)\right|\leq\frac{\tau^3}{1-\tau L_1}\left[\tau^{m-3}\cdot L^m\cdot\left\|\varphi_0-\varphi_1\right\|_{C[0,T]}+\frac{L'}{12n^2}\right],$$

$\forall m\in\mathbb{N}^*,\ \forall k=\overline{n+1,q}$.

Proof. Follows from Remark 2, Proposition 8 and Theorem 5. $\square$

EXAMPLE 10. To illustrate the method we consider the following numerical example: Let $\tau=2$, $l=3$, $T=6$, $n=4$, $q=16$, $w=0.39745735$,

$$\varphi(t)=w\cdot\frac{2t+5}{5t+15}$$

and

$$f(t,x)=\frac{1}{8}\cdot\arctan^2(t+x).$$

We use the following stop criterion: for chosen $\varepsilon>0$ find the smallest $m\in\mathbb{N}^*$ such that

$$\left|\widetilde{\varphi_m}(t_k)-\widetilde{\varphi_{m-1}}(t_k)\right|<\varepsilon,\qquad \forall k=\overline{n+1,q}.$$

For $\varepsilon=10^{-6}$ we found $m=6$ and in the bellow table there are $\widetilde{\varphi_m}(t_k)$ and $\widetilde{\varphi_{m-1}}(t_k)$ for $k=\overline{0,q-1}$.

Table 1. Numerical results

t_k	$\widetilde{\varphi_m}(t_k)$	$\widetilde{\varphi_{m-1}}(t_k)$	t_k	$\widetilde{\varphi_m}(t_k)$	$\widetilde{\varphi_{m-1}}(t_k)$
t_0	0.07949147	0.07949147	t_8	0.16369548	0.16369544
t_1	0.10598863	0.10598863	t_9	0.24161442	0.24161434
t_2	0.1192372	0.1192372	t_{10}	0.30979356	0.30979345
t_3	0.12718635	0.12718635	t_{11}	0.36226913	0.36226901
t_4	0.13248578	0.13248578	t_{12}	0.40083609	0.40083597
t_5	0.078338039	0.078338038	t_{13}	0.42922205	0.42922195
t_6	0.063332751	0.063332746	t_{14}	0.4506688	0.45066872
t_7	0.095814042	0.095814023	t_{15}	0.46742765	0.4674276.

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REFERENCES

[1] CERONE, P., DRAGOMIR, S.S., *Trapeziodal and Midpoint-Type Rules from Inequalities Point of View* (ch. 3 and 4 in Handbook of "Analytic Computational Methods in Applied Mathematics", ed. G. Anastassiou), Chapman&Hall/CRC, New York, 2000.

[2] CERONE, P. DRAGOMIR, S.S., *Three point quadrature rules involving, at most, at first derivative*, RGMIA (preprint), **2**, no. 4, pp. 511–583, 1999.

[3] GUO, D. LAKSHMIKANTHAM, V., *Positive solutions of nonlinear integral equations arising in infectious diseases*, J. Math. Anal. Appl., **134**, pp. 1–8, 1988.

[4] IANCU, C., *A numerical method for approximating the solution of an integral equation from biomathematics*, Studia Univ. "Babeş-Bolyai", Mathematica, **XLIII**, no. 4, pp. 37–45, 1998.

[5] PRECUP, R., *Positive solutions of the initial value problem for an integral equation modelling infectious disease*, Seminar on Fixed Point Theory, Preprint no. 3, ed. I.A. Rus, pp. 25–30, 1991.

[6] PRECUP, R., *Periodic solutions for an integral equation from biomathematics via Leray-Schauder principle*, Studia Univ. "Babeş-Bolyai", Mathematica, **XXXIX**, no. 1, pp. 47–58, 1994.

[7] PRECUP, R., *Monotone technique to the initial values problem for a delay integral equation from biomathematics*, Studia Univ. "Babeş-Bolyai", Mathematica, **XL**, no. 2, pp. 63–73, 1995.

[8] RUS, I. A., *A delay integral equation from biomathematics*, Seminar of Differential Equations, Preprint, no. 3, pp.87–90, 1989.

[9] RUS, I. A., *Differential equations, integral equations and dynamical systems*, Transilvania Press, Cluj Napoca, 1996 (in Romanian).

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