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BIERMANN INTERPOLATION OF BIRKHOFF TYPE

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Abstract. If $P_0, P_1, ..., P_r$ and $Q_0, Q_1, ..., Q_r$ are Birkhoff univariate projectors which form the chains

$$P_0 \leq P_1 \leq \cdots \leq P_r, \quad Q_0 \leq Q_1 \leq \cdots \leq Q_r,$$

we can define the Biermann operator of Birkhoff type

$$B_r^B = P_0'Q_r'' \oplus P_1'Q_{r-1}'' \oplus \cdots \oplus P_r'Q_0'',$$

where $P_1', \ldots, P_r'$, $Q_1'', \ldots, Q_r''$ are the parametric extension.

We give the representations of Biermann interpolant of Birkhoff type for two particular cases (Abel-Goncharov and Lidstone projectors) and we calculate the approximation order of Biermann interpolant in these cases.

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1. PRELIMINARIES

Let X, Y be the linear space on R or C. The linear operator P defined on space X is called projector if $P^2 = P$. The operator $P^c = I - P$, where I is identity operator, is called the remainder projector of P. If P is projector on space X then the range space of projector P is denoted by

$$\mathcal{R}(P) = \{Pf|\ f \in X\}. \tag{1}$$

The set of interpolation points of projector P is denoted by $\mathcal{P}(P)$.

PROPOSITION 1. *If P, Q are comutative projectors then*

$$\begin{aligned} \mathcal{R}(P \oplus Q) &= \mathcal{R}(P) + \mathcal{R}(Q), \\ \mathcal{P}(P \oplus Q) &= \mathcal{P}(P) \cup \mathcal{P}(Q). \end{aligned} \tag{2}$$

If P_1 and P_2 are projectors on space X, we define relation "$\leq$" by

$$P_1 \leq P_2 \Leftrightarrow P_1P_2 = P_1.$$

Let $f \in C(X \times Y)$ and $x \in X$. We define $f^x \in C(Y)$ by

$$f^x(t) = f(x,t), \quad t \in Y.$$

For $y \in Y$ we define ${}^y f \in C(X)$ by

$$^y f(s) = f(s,y), \quad s \in X.$$

Let P be a linear and bounded operator on $C(X)$. The parametric extension P' of P is defined by

$$(P'f)(x,y) = (P^y f)(x). \tag{3}$$

If P is a linear and bounded operator on $C(Y)$ then the parametric extension Q'' of Q is defined by

$$(Q''f)(x,y) = (Qf^x)(y). \tag{4}$$

Proposition 2. *Let $r \in \mathbb{N}$, $P_0, P_1, \ldots, P_r$ be univariate interpolation projectors on $C(X)$ and $Q_0, Q_1, \ldots, Q_r$ univariate interpolation projectors on $C(Y)$. Let $P'_0, \ldots, P'_r$, $Q''_0, \ldots, Q''_r$ be the corresponding parametric extension. We assume that*

$$P_0 \le P_1 \le \cdots \le P_r, \quad Q_0 \le Q_1 \le \cdots \le Q_r. \tag{5}$$

Then

$$B_r = P'_0 Q''_r \oplus P'_1 Q''_{r-1} \oplus \cdots \oplus P'_r Q''_0 \tag{6}$$

is projector and it has representation

$$B_r = \sum_{m=0}^{r} P'_m Q''_{r-m} - \sum_{m=0}^{r-1} P'_m Q''_{r-m-1}. \tag{7}$$

Moreover, we have

$$B_r^c = P_r'^c + P_{r-1}'^c Q_0''^c + \cdots + P_0'^c Q_{r-1}''^c + Q_r''^c - (P_r'^c Q_0''^c + \cdots + P_0'^c Q_r''^c), \tag{8}$$

where $P^c = I - P$, I the identity operator.

For the proof of Proposition 2 one can see [6].

Remark 3. If P_i, $i = \overline{0,r}$, and Q_j, $j = \overline{0,r}$, are Lagrange interpolation projectors which form the chains with respect to relation "$\le$", the projector B_r is called Biermann interpolation projector (see [6]). In [10] we instead the Lagrange projectors by Hermite projectors. In this article, our objective is to construct chains of Birkhoff interpolation projectors and, with their aid, the Biermann interpolant of Birkhoff type.

2. MAIN RESULT

Let be the univariate projectors of Birkhoff interpolation

$$P_0, \ldots, P_r, Q_0, \ldots, Q_r$$

given by relations

$$(P_m f)(x) = \sum_{i=0}^{k_m} \sum_{p \in I_{i,m}} f^{(p)}(x_i) b_{ip}^m(x), \quad 0 \le m \le r, \tag{9}$$

$$(Q_n g)(y) = \sum_{j=0}^{l_n} \sum_{q \in J_{j,n}} g^{(q)}(y_j) \widetilde{b}_{jq}^n(y), \quad 0 \le n \le r.$$

Assume that

$$\{x_0, \dots, x_{k_r}\} \subseteq [a,b], \quad \{y_0, \dots, y_{l_r}\} \subseteq [c,d]$$

with

$$0 \le k_0 \le k_1 \le \cdots \le k_r, \quad 0 \le l_0 \le l_1 \le \cdots \le l_r \tag{10}$$

and

$$\begin{aligned} I_{i,m} &\subseteq I_{i,m+1},\ i = \overline{0,k_m},\ m = \overline{0,r-1}, \\ J_{j,n} &\subseteq J_{j,n+1},\ j = \overline{0,l_n},\ n = \overline{0,r-1}. \end{aligned} \tag{11}$$

The cardinal functions b_{ip}^m, $m = \overline{0,r}$, and $\widetilde{b}_{jq}^n$, $n = \overline{0,r}$, satisfy

$$\begin{cases} b_{ip}^{m\ (j)}(x_\nu) = 0,\ \nu \ne i,\ j \in I_{\nu,m} \\ b_{ip}^{m\ (j)}(x_i) = \delta_{jp},\ j \in I_{i,m} \end{cases}$$

for $p \in I_{i,m}$, $\nu, i = \overline{0,k_m}$ and, respectively,

$$\begin{cases} \widetilde{b}_{jq}^{n\ (i)}(y_\nu) = 0,\ \nu \ne j,\ i \in J_{\nu,n} \\ \widetilde{b}_{jq}^{n\ (i)}(y_j) = \delta_{iq},\ i \in J_{j,n} \end{cases}$$

for $q \in J_{j,n}$, $\nu, j = \overline{0,l_n}$.

THEOREM 4. *The parametric extensions*

$$P_0', \dots, P_r', \quad Q_0'', \dots, Q_r''$$

are bivariate interpolation projectors which form the chains

$$P_0' \le \cdots \le P_r', \quad Q_0'' \le \cdots \le Q_r''.$$

Proof. Let be $0 \le m_1 \le m_2 \le r$. Then

$$\begin{aligned} k_{m_1} &\le k_{m_2}, \\ I_{i,m_1} &\subseteq I_{i,m_2},\ i \le k_{m_1}, \end{aligned} \tag{12}$$

We have that

$$\begin{aligned} &(P_{m_1}' P_{m_2}' f)(x,y) = \\ &= \sum_{i_1=0}^{k_{m_1}} \sum_{p_1 \in I_{i_1,m_1}} \left(\sum_{i_2=0}^{k_{m_2}} \sum_{p_2 \in I_{i_2 m_2}} f^{(p_2,0)}(x_{i_2}, y) b_{i_2 p_2}^{m_2\ (p_1)}(x_{i_1}) \right) b_{i_1 p_1}^{m_1}(x). \end{aligned} \tag{13}$$

But

$$b_{i_2 p_2}^{m_2\ (p_1)}(x_{i_1}) = \delta_{i_1 i_2} \delta_{p_1 p_2}. \tag{14}$$

From (12), (13) and (14) we have

$$(P_{m_1}' P_{m_2}' f)(x,y) = \sum_{i_1=0}^{k_{m_1}} \sum_{p_1 \in I_{i_1,m_1}} f^{(p_1,0)}(x_{i_1}, y) b_{i_1 p_1}^{m_1}(x) = (P_{m_1}' f)(x,y),$$

i.e.,

$$P'_{m_1} \le P'_{m_2}.$$

Thus $P'_0, P'_1, ..., P'_r$ form a chain. Analogous $Q''_0, Q''_1, ..., Q''_r$ are projectors which form a chain. $\square$

Moreover, we have

$$P'_m Q''_n = Q''_n P'_m, \quad 0 \le m, n \le r.$$

The tensor product projector $P'_m Q''_n$ of bivariate interpolation has representation

$$(P'_m Q''_n f)(x,y) = \sum_{i=0}^{k_m} \sum_{p \in I_{i,m}} \sum_{j=0}^{l_n} \sum_{q \in J_{j,n}} f^{(p,q)}(x_i, y_j) h_{ip}^m(x) \widetilde{h}_{jq}^n(y)$$

and it has the interpolation properties

$$(P'_m Q''_n f)^{(p,q)}(x_i, y_j) = f^{(p,q)}(x_i, y_j), \quad 0 \le i \le k_m, \quad 0 \le j \le l_n, \\ p \in I_{i,m}, \quad q \in J_{j,n}.$$

The projectors $P'_0, \ldots, P'_r$, $Q''_0, \ldots, Q''_r$ generate a distributive lattice ξ of projectors on $C([a,b] \times [c,d])$. A special element in this lattice is

$$B_r^B = P'_0 Q''_r \oplus P'_1 Q''_{r-1} \oplus \cdots \oplus P'_r Q''_0, \quad r \in \mathbb{N},$$

called Biermann projector of Birkhoff type and which has the interpolation properties

$$(B_r^B f)^{(p,q)}(x_i, y_j) = f^{(p,q)}(x_i, y_j), \ 0 \le i \le k_m, \ 0 \le j \le l_{r-m}, \quad 0 \le m \le r, \\ p \in I_{i,m} \backslash I_{i,m-1}, q \in J_{j,r-m} \tag{15}$$

with $I_{i,s} = \emptyset$, $k_s < i \le k_{s+1}$, $s = \overline{-1, r-1}$ with $k_{-1} = -1$.

Let $\alpha_i = |I_{0,i}| + \cdots + |I_{k_i,i}|$, $\beta_i = |J_{0,i}| + \cdots + |J_{l_i,i}|$, $0 \le i \le r$. The range space of projector B_r^B is

$$\mathcal{R}(B_r^H) = \Pi_{\alpha_0 - 1} \otimes \Pi_{\beta_r - 1} + \cdots + \Pi_{\alpha_r - 1} \otimes \Pi_{\beta_0 - 1}. \tag{16}$$

The properties (15) and (16) result using Proposition 1.

Let be $h = b - a = d - c$ and $q = \min\{\alpha_{r-m-1} + \beta_m, \ -1 \le m \le r\}$ with $\alpha_{-1} = 0$, $\beta_{-1} = 0$. From (8) we have

$$f(x,y) - (B_r^B f)(x,y) = \mathcal{O}(h^q), \quad h \to 0. \tag{17}$$

3. PARTICULAR CASES

3.1. Biermann interpolation of Abel-Goncharov type. Let be the univariate Abel-Goncharov interpolation projectors

$$(P_m f)(x) = \sum_{i=0}^{m} g_{i,m}(x) f^{(i)}(x_i), \quad m = \overline{0,r},$$
$$(Q_n f)(y) = \sum_{j=0}^{n} \widetilde{g}_{j,n}(y) f^{(j)}(y_j), \quad n = \overline{0,r},$$

i.e.,

$$\begin{aligned} k_m &= m, \quad m = \overline{0,r}, \\ l_n &= n, \quad n = \overline{0,r}, \\ I_{j,i} &= \{j\}, \quad j = \overline{0,r}, i = \overline{j,r}, \\ J_{i,j} &= \{i\}, \quad i = \overline{0,r}, j = \overline{i,r}. \end{aligned}$$

According to [5], the cardinal functions $g_{i,m}$, $i = \overline{0,m}$, are given by recurrence relations

$$\begin{aligned} &g_{0,m}(x) = 1, \\ &g_{1,m}(x) = 1 - x_0, \\ &g_{i,m}(x) = \tfrac{1}{i!}\Big(x^i - \sum_{j=0}^{i-1} g_{i,m}(x)\tbinom{i}{j} x_j^{i-j}\Big), \;\; i = \overline{2,m}, \end{aligned}$$

for $m = \overline{0,r}$.

One remark that the cardinal functions $g_{i,m}$ depend only the nodes $x_0, x_1, \ldots, x_{i-1}$. It follows that the functions $g_{i,m}$, $i = \overline{0,r}$, are the same for $m = \overline{i,r}$. One denotes

$$g_i(x) := g_{i,m}(x), \quad i = \overline{0,r}, \; m = \overline{i,r}.$$

We have analogous relations for the cardinal functions $\widetilde{g}_{j,n}$ and we denote

$$\widetilde{g}_j(x) := \widetilde{g}_{j,m}(x), \quad j = \overline{0,r}, \; m = \overline{j,r}.$$

The indices sets of derivatives satisfy the chains of equalities

$$I_{j,j} = I_{j,j+1} = \ldots = I_{j,r}, \quad j = \overline{0,r},$$

$$J_{i,i} = J_{i,i+1} = \ldots = J_{i,r}, \quad i = \overline{0,r}.$$

It follows that relations (11) hold.

We can define the Biermann operator of Abel-Goncharov type

$$B_r^{AG} = P_0' Q_r'' \oplus P_1' Q_{r-1}'' \oplus \ldots \oplus P_r' Q_0'', \tag{18}$$

where P_i', $i = \overline{0,r}$, and Q_j'', $j = \overline{0,r}$, are the parametric extensions.

Taking into account (7) we obtain the following representation for Biermann interpolant B_r^{AG}

$$(B_r^{AG}f)(x,y) = \sum_{m=0}^{r}(P'_m(Q''_{r-m} - Q''_{r-m-1})f)(x,y) = \sum_{m=0}^{r}\sum_{i=0}^{m} g_i(x)\widetilde{g}_{r-m}(y)f^{(i,r-m)}(x_i,y_{r-m}), \tag{19}$$

where $Q''_{-1}=0$.

Using Proposition 1, the Biermann interpolant B_r^{AG} has the interpolation properties

$$(B_r^{AG}f)^{(i,r-j)}(x_i,y_{r-j}) = f^{(i,r-j)}(x_i,y_{r-j}), \quad i=\overline{0,r}, \quad j=\overline{0,i}. \tag{20}$$

Remark 5. The set of nodes form a triungular grid

$$\begin{array}{lllll} (x_0,y_r) & & & & \\ (x_0,y_{r-1}), & (x_1,y_{r-1}) & & & \\ \ldots & & & & \\ (x_0,y_1), & (x_1,y_1), & \ldots & (x_{r-1},y_1) & \\ (x_0,y_0), & (x_1,y_0), & \ldots & (x_{r-1},y_0), & (x_r,y_0) \end{array} \quad \square$$

Let $h=b-a=d-c$. According to

$$\|f(x)-P_mf(x)\| = O(h^{m+1}), \quad \text{if } f\in C^{m+1}[a,b],$$
$$\|f(y)-Q_nf(x)\| = O(h^{n+1}), \quad \text{if } \ f\in C^{n+1}[c,d],$$

and relation (8), the approximation order of Biermann interpolant of Abel-Goncharov type is $r+1$, i.e.,

$$\left\|f(x,y)-B_r^{AG}f(x,y)\right\| = O(h^{r+1}), \quad \text{if} \quad f\in C^{r+1}([a,b]\times[c,d]). \tag{21}$$

Remark 6. The approximation order of Biermann operator B_r^{AG} is the same with the approximation order of tensor product operator $P'_rQ''_r$. If P is an interpolation projector we denote by $I_P(f)$ the set of data about f (values of function f and/or certain of its partial derivatives at nodes). We have

$$|I_{B_r^{AG}}(f)| = \tfrac{(r+1)(r+2)}{2},$$
$$|I_{P'_rQ''_r}(f)| = (r+1)^2,$$

and

$$I_{B_r^{AG}}(f) \subset I_{P'_rQ''_r}(f).$$

It follows that the Biermann operator B_r^{AG} is more efficient than tensor product operator $P'_rQ''_r$. $\square$

3.2. Biermann interpolation of Lidstone type. Let be the univariate Lidstone interpolation projectors

$$(P_m f)(x) = \sum_{i=0}^{m} [l_{0,i}^m(x) f^{(2i)}(x_0) + l_{1,i}^m(x) f^{(2i)}(x_1)], \quad m = \overline{0,r},$$
$$(Q_n g)(y) = \sum_{j=0}^{n} [\widetilde{l}_{0,j}^n(y) f^{(2j)}(y_0) + \widetilde{l}_{1,j}^n(y) f^{(2j)}(y_1)], \quad n = \overline{0,r},$$

i.e.,

$$\begin{aligned} k_0 &= k_1 = \ldots = k_r = 1, \\ l_0 &= l_1 = \ldots = l_r = 1, \\ I_{0,i} &= I_{1,i} = \{0, 2, \ldots, 2i\}, \quad i = \overline{0,r}, \\ J_{0,j} &= J_{1,j} = \{0, 2, \ldots, 2j\}, \quad j = \overline{0,r}. \end{aligned}$$

If we denote $h = x_1 - x_0 = y_1 - y_0$, from [1] we can obtain cardinal functions by relations

$$\begin{aligned} l_{0,i}^m(x) &= \Delta_i(\tfrac{x_1 - x}{h}) h^{2i}, \\ l_{1,i}^m(x) &= \Delta_i(\tfrac{x - x_0}{h}) h^{2i}, \end{aligned}$$

$m = \overline{0,r}$, $i = \overline{0,m}$, where the functions Δ_i, $i = \overline{0,r}$, are given by recurrence relations

$$\begin{aligned} \Delta_0(x) &= x, \\ \Delta_n(x) &= \int_0^1 g_n(x,s) s \mathrm{d}s, \quad n \geq 1, \end{aligned}$$

with

$$\begin{aligned} g_1(x,s) &= \begin{cases} (x-1)s, & s \leq x, \\ (s-1)x, & x \leq s, \end{cases} \\ g_n(x,s) &= \int_0^1 g_1(x,t) g_{n-1}(t,s) \mathrm{d}t, \quad n \geq 2. \end{aligned}$$

One remark that the cardinal function $l_{0,i}^m$ and $l_{1,i}^m$ don't depend of m. We can make notation

$$\begin{aligned} l_{0,i}(x) &:= l_{0,i}^m(x), \quad i = \overline{0,r},\ m = \overline{i,r}, \\ l_{1,i}(x) &:= l_{1,i}^m(x), \quad i = \overline{0,r},\ m = \overline{i,r}. \end{aligned}$$

We have analogous relations for cardinal functions $\widetilde{l}_{0,j}^n$, $\widetilde{l}_{1,j}^n$ and we denote

$$\begin{aligned} \widetilde{l}_{0,j}(y) &:= \widetilde{l}_{0,j}^n(y), \quad j = \overline{0,r},\ n = \overline{j,r}, \\ \widetilde{l}_{1,j}(y) &:= \widetilde{l}_{1,j}^n(y), \quad j = \overline{0,r},\ n = \overline{j,r}. \end{aligned}$$

The indices sets of derivatives verify relations (11) because

$$I_{j,i}=\{0,2,\ldots,2i\}\subset\{0,2,\ldots,2i,2i+2\}=I_{j,i+1},\quad j=0,1,\ i=\overline{0,r-1},$$
$$J_{i,j}=\{0,2,\ldots,2j\}\subset\{0,2,\ldots,2j,2j+2\}=I_{i,j+1},\quad i=0,1,\ j=\overline{0,r-1}.$$

We can define the Biermann operator of Lidstone type

$$B_r^L=P_0'Q_r''\oplus P_1'Q_{r-1}''\oplus\ldots\oplus P_r'Q_0'', \tag{22}$$

where $P_i',\ i=\overline{0,r}$, and $Q_j'',\ j=\overline{0,r}$, are the parametric extensions.

From (7), we obtain the following representation for Biermann operator B_r^L

(23)

$$\begin{aligned}(B_r^Lf)(x,y)&=\\ =&\sum_{m=0}^{r}(P_m'(Q_{r-m}''-Q_{r-m-1}'')f)(x,y)\\ =&\sum_{m=0}^{r}\sum_{i=0}^{m}l_{0,i}(x)[\widetilde{l}_{0,2r-2m}(y)f^{(2i,2r-2m)}(x_0,y_0)+\widetilde{l}_{1,2r-2m}(y)f^{(2i,2r-2m)}(x_0,y_1)]\\ +&\sum_{m=0}^{r}\sum_{i=0}^{m}l_{1,i}(x)[\widetilde{l}_{0,2r-2m}(y)f^{(2i,2r-2m)}(x_1,y_0)+\widetilde{l}_{1,2r-2m}(y)f^{(2i,2r-2m)}(x_1,y_1)],\end{aligned}$$

where $Q_{-1}''=0$.

Using Proposition 1, it folows that the Biermann projector B_r^L has the interpolations properties

(24)

$$(B_r^Lf)^{(2i,2(r-j))}(x_k,y_l)=f^{(2i,2(r-j))}(x_k,y_l),\quad i=\overline{0,r},\ j=\overline{0,i},\quad k,l\in\{0,1\}.$$

From [1] we have

$$\|f(x)-P_mf(x)\|=O(h^{2m}),\qquad \text{if } f\in C^{2m}[a,b],$$
$$\|g(y)-Q_ng(y)\|=O(h^{2n}),\qquad \text{if } g\in C^{2n}[c,d].$$

Taking into account (8), the approximation order of Biermann operator B_r^L is $2r$, i.e.

$$\left\|f(x,y)-B_r^Lf(x,y)\right\|=O(h^{2r}),\qquad \text{if } f\in C^{2r}([a,b]\times[c,d]). \tag{25}$$

Remark 7. The approximation order of Biermann operator B_r^L is the same with the approximation order of tensor product operator $P_r'Q_r''$. We have

$$|I_{B_r^L}(f)|=2(r+1)(r+2),$$
$$|I_{P_r'Q_r''}(f)|=4(r+1)^2$$

and

$$I_{B_r^L}(f)\subset I_{P_r'Q_r''}(f),$$

where $I_P(f)$ is the set of data about f used by interpolation projector P. It follows that the Biermann operator B_r^B is more efficient than tensor product operator $P_r'Q_r''$. $\square$

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