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BOUNDS FOR THE REMAINDER IN THE BIVARIATE SHEPARD INTERPOLATION OF LIDSTONE TYPE

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Abstract. We study the bivariate Shepard-Lidstone interpolation operator and obtain new estimates for the remainder. Some numerical examples are provided.

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Keywords. Bivariate Shepard-Lidstone interpolation, remainder.

1. INTRODUCTION

As it is pointed out in [7] and [15], interpolation at nodes having no exploitable pattern is referred to as the case of scattered data and there are two important methods of interpolation in this case: the method of Shepard and the interpolation by radial basis functions.

Consider $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$ and let $\Delta : a = x_0 < x_1 < \ldots < x_{N+1} = b$ and $\Delta' : c = y_0 < y_1 < \ldots < y_{M+1} = d$ denote uniform partitions of the intervals $[a, b]$ and $[c, d]$ with stepsizes $h = (b-a)/(N+1)$ and $l = (d-c)/(M+1)$, respectively. Further, let $\rho = \Delta \times \Delta'$ be a rectangular partition of $[a, b] \times [c, d]$. For the univariate function f and the bivariate function g and each positive integer r we denote by $D^r f = \mathrm{d}^r f / \mathrm{d}\, x^r$, $D_x^r g = \partial^r g/\partial x^r$ and $D_y^r g = \partial^r g/\partial y^r$.

According to [1] and [2], for a fixed Δ denote the set $L_m(\Delta) = \{h \in C[a, b] :$ h is a polynomial of degree at most $2m-1$ in each subinterval $[x_i, x_{i+1}]$, $0 \leq i \leq N\}$.

Definition 1. [2] *For a given function $f \in C^{2m-2}[a, b]$ we say that $L_m^\Delta f$ is the Lidstone interpolant of f if $L_m^\Delta f \in L_m(\Delta)$ with*

$$D^{2k}(L_m^\Delta f)(x_i) = f^{(2k)}(x_i), \quad 0 \leq k \leq m-1,\ 0 \leq i \leq N+1.$$

According to [2], for $f \in C^{2m-2}[a, b]$ the Lidstone interpolant $L_m^\Delta f$ uniquely exists and on the subinterval $[x_i, x_{i+1}]$, $0 \leq i \leq N$, can be explicitly expressed

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as
(1)
$$(L_m^\Delta f)|_{[x_i,x_{i+1}]}(x) = \sum_{k=0}^{m-1} \left[\Lambda_k\left(\tfrac{x_{i+1}-x}{h}\right) f^{(2k)}(x_i) + \Lambda_k\left(\tfrac{x-x_i}{h}\right) f^{(2k)}(x_{i+1})\right] h^{2k},$$

where Λ_k is the Lidstone polynomial of degree $2k+1$, $k \in \mathbb{N}$ on the interval $[0,1]$.

We have the interpolation formula

$$f = L_m^\Delta f + R_m^\Delta f,$$

where $R_m^\Delta f$ denotes the remainder.

For a fixed rectangular partition $\rho = \Delta \times \Delta'$ of $[a,b] \times [c,d]$ the set $L_m(\rho)$ is defined as $L_m(\rho) = L_m(\Delta) \otimes L_m(\Delta')$ (see, e.g., [1] and [2]).

Definition 2. [2] *For a given function $f \in C^{2m-2,2m-2}([a,b]\times[c,d])$ we say that $L_m^\rho f$ is the two-dimensional Lidstone interpolant of f if $L_m^\rho f \in L_m(\rho)$ with*

$$D_x^{2\mu} D_y^{2\nu}(L_m^\rho f)(x_i,y_j) = f^{(2\mu,2\nu)}(x_i,y_j), \quad 0 \le i \le N+1,\ 0 \le j \le M+1,$$
$$0 \le \mu,\nu \le m-1.$$

According to [2], for $f \in C^{2m-2,2m-2}([a,b]\times[c,d])$, the Lidstone interpolant $L_m^\rho f$ uniquely exists and can be explicitly expressed as

$$(L_m^\rho f)(x,y) = \sum_{i=0}^{N+1}\sum_{\mu=0}^{m-1}\sum_{j=0}^{M+1}\sum_{\nu=0}^{m-1} r_{m,i,\mu}(x) r_{m,j,\nu}(y) f^{(2\mu,2\nu)}(x_i,y_j), \tag{2}$$

where $r_{m,i,j}$, $0 \le i \le N+1$, $0 \le j \le m-1$, are the basic elements of $L_m(\rho)$ satisfying

$$D^{2\upsilon} r_{m,i,j}(x_\mu) = \delta_{i\mu}\delta_{2\upsilon,j}, \quad 0 \le \mu \le N+1,\ 0 \le \upsilon \le m-1. \tag{3}$$

Lemma 3. [2] *If $f \in C^{2m-2,2m-2}([a,b]\times[c,d])$, then*

$$(L_m^\rho f)(x,y) = (L_m^\Delta L_m^{\Delta'} f)(x,y) = (L_m^{\Delta'} L_m^\Delta f)(x,y).$$

Corollary 4. [2] *For a function $f \in C^{2m-2,2m-2}([a,b]\times[c,d])$, from Lemma 3, we have that*

$$\begin{aligned} f - L_m^\rho f &= (f - L_m^\Delta f) + L_m^\Delta(f - L_m^{\Delta'} f) \\ &= (f - L_m^\Delta f) + [L_m^\Delta(f - L_m^{\Delta'} f) - (f - L_m^{\Delta'} f)] + (f - L_m^{\Delta'} f). \end{aligned} \tag{4}$$

With the previous assumptions we denote by $L_m^{\Delta,i} f$ the restriction of the Lidstone interpolation polynomial $L_m^\Delta f$ to the subinterval $[x_i,x_{i+1}]$, $0 \le i \le N$, given by (1), and in analogous way we obtain the expression of $L_m^{\Delta',i} f$, the restriction of $L_m^{\Delta'} f$, to the subinterval $[y_i,y_{i+1}] \subseteq [c,d]$, $0 \le i \le N$. We denote by S_L the univariate combined Shepard-Lidstone operator, introduced by us in [4]:

$$(S_L f)(x) = \sum_{i=0}^{N} A_i(x)(L_m^{\Delta,i} f)(x),$$

with A_i, $i = 0, ..., N$, given by

$$A_i\,(x,y) = \prod_{\substack{j=0\\ j\neq i}}^{N} r_j^{\mu}\,(x,y) \Big/ \Bigg(\sum_{k=0}^{N} \prod_{\substack{j=0\\ j\neq k}}^{N} r_j^{\mu}\,(x,y) \Bigg) \tag{5}$$

and

$$\sum_{i=0}^{N} A_i = 1. \tag{6}$$

The univariate Shepard-Lidstone interpolation formula is

$$f = S_L f + R_L f. \tag{7}$$

We consider $f \in C^{2m-2,2m-2}([a,b] \times [c,d])$ and the set of Lidstone functionals

$$\Lambda_{Li}^{i} = \Big\{ f(x_i, y_i), f(x_{i+1}, y_{i+1}), \ldots, f^{(2m-2,2m-2)}(x_i, y_i), \\ f^{(2m-2,2m-2)}(x_{i+1}, y_{i+1}) \Big\},$$

regarding each subrectangle $[x_i, x_{i+1}] \times [y_i, y_{i+1}], 0 \le i \le N$, with $\left|\Lambda_{Li}^{i}\right| = 4m,\ 0 \le i \le N$. We denote by $L_m^{\rho,i} f$ the restriction of the polynomial given by (2) to the subrectangle $[x_i, x_{i+1}] \times [y_i, y_{i+1}], 0 \le i \le N$. This $2m-1$ polynomial, in each variable, solves the interpolation problem corresponding to the set Λ_{Li}^{i}, $0 \le i \le N$ and it uniquely exists.

We have

$$(L_m^{\rho,i} f)^{(2\nu,2\nu)}(x_k, y_k) = f^{(2\nu,2\nu)}(x_k, y_k),$$

$0 \le i \le N;\ 0 \le \nu \le m-1;\ k = i, i+1.$

The bivariate Shepard operator of Lidstone type S^{Li}, introduced by us in [5], is given by

$$(S^{Li} f)(x,y) = \sum_{i=0}^{N} A_i(x,y)(L_m^{\rho,i} f)(x,y). \tag{8}$$

We obtain the bivariate Shepard-Lidstone interpolation formula,

$$f = S^{Li} f + R^{Li} f, \tag{9}$$

where $S^{Li} f$ is given by (8) and $R^{Li} f$ denotes the remainder of the interpolation formula.

Next, we give an error estimation using the modulus of smoothness of order k. For a function g defined on $[a,b]$ we have

$$\omega_k(g;\delta) = \sup\{|\Delta_h^k g(x)|\ : |h|\ \le \delta,\ x, x+kh \in [a,b]\},$$

with $\delta \in [0,(b-a)/k]$ and

$$\Delta_h^k g(x) = \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} g(x+ih).$$

We consider the norm in the set $C(X)$ of continuous functions defined on X by

$$\|f\|_{C(X)} = \max_{x\in X} |f(x)|.$$

We recall first a result from [17]:

THEOREM 5. [17] *Let L be a bounded operator and let $L(P) = P$ for every $P \in \mathbb{P}_{k-1}$. Then for every bounded function $f : [a,b] \to \mathbb{R}$ the following inequality is fulfilled*

$$\|f - L(f)\|_{C[a,b]} \le (1 + \|L\|_{[a,b]}) W_k \omega_k \left(f; \tfrac{b-a}{k}\right),$$

where W_k is Whitney's constant.

We apply this result for the operators L_m^{Δ} and $L_m^{\Delta'}$. For $f \in C[a,b]$ and $g \in C[c,d]$ we have

$$\begin{aligned} &\|f - L_m^{\Delta}(f)\|_{C[a,b]} \le (1 + \|L_m^{\Delta}\|_{[a,b]}) W_{2m}\omega_{2m} \left(f; \tfrac{b-a}{2m}\right), \\ &\|g - L_m^{\Delta'}(g)\|_{C[c,d]} \le (1 + \|L_m^{\Delta'}\|_{[c,d]}) W_{2m}\omega_{2m} \left(g; \tfrac{d-c}{2m}\right). \end{aligned} \tag{10}$$

Now we give an estimation of the remainder $R_L f$ from (7), in terms of the modulus of smoothness.

THEOREM 6. *If $f \in C^{2m-2}[a,b]$, then*

$$\|R_L f\|_{C[a,b]} \le (1 + \left\|L_m^{\Delta}\right\|_{[a,b]}) W_{2m}\omega_{2m} \left(f; \tfrac{b-a}{2m}\right). \tag{11}$$

Proof. We have

$$\begin{aligned} (R_L f)(x) &= f(x) - \sum_{i=0}^{N} A_i(x)(L_m^{\Delta,i} f)(x) \\ &= \sum_{i=0}^{N} A_i(x) f(x) - \sum_{i=0}^{N} A_i(x)(L_m^{\Delta,i} f)(x) \\ &= \sum_{i=0}^{N} A_i(x)[f(x) - (L_m^{\Delta,i} f)(x)], \end{aligned}$$

and taking into account (10) and that

$$\sum_{i=0}^{N} |A_i(x)| = 1,$$

relation (11) follows. □

The next result provides an estimation of the error in formula (9).

THEOREM 7. *If* $f \in C^{2m-2,2m-2}([a,b]\times[c,d])$, *then*

$$\begin{aligned}\left\|R^{Li}f\right\|_{C[a,b]} \leq & (1+\left\|L_m^{\Delta}\right\|_{C[a,b]})W_{2m}\max_{y\in[c,d]}\omega_{2m}\left(f(\cdot,y);\tfrac{b-a}{2m}\right)\\ &+(1+\left\|L_m^{\Delta}\right\|_{C[a,b]})W_{2m}\max_{y\in[c,d]}\omega_{2m}((f-L_m^{\Delta'}f)(\cdot,y);\tfrac{b-a}{2m})\\ &+(1+\|L_m^{\Delta'}\|_{C[c,d]})W_{2m}\max_{x\in[a,b]}\omega_{2m}\left(f(x,\cdot);\tfrac{d-c}{2m}\right),\end{aligned}$$

where W_k *is Whitney's constant.*

Proof. Taking into account (8) and (6) we get

$$\begin{aligned}(R^{Li}f)(x,y) &= f(x,y)-(S^{Li}f)(x,y)\\ &= f(x,y)-\sum_{i=0}^{N}A_i(x,y)(L_m^{\rho,i}f)(x,y)\\ &= \sum_{i=0}^{N}A_i(x,y)f(x,y)-\sum_{i=0}^{N}A_i(x,y)(L_m^{\rho,i}f)(x,y)\\ &= \sum_{i=0}^{N}A_i(x,y)[f(x,y)-(L_m^{\rho,i}f)(x,y)].\end{aligned}$$

Next applying the results (4) given by Corollary 4 and (6) it follows that

$$\begin{aligned}(R^{Li}f)(x,y) =&\sum_{i=0}^{N}A_i(x,y)\{(f-L_m^{\Delta,i}f)(x,y)\\ &+[L_m^{\Delta,i}(f-L_m^{\Delta',i}f)(x,y)-(f-L_m^{\Delta',i}f)(x,y)]\\ &+(f-L_m^{\Delta',i}f)(x,y)\}\\ =&\sum_{i=0}^{N}A_i(x,y)(f-L_m^{\Delta,i}f)(x,y)\\ &+\sum_{i=0}^{N}A_i(x,y)[L_m^{\Delta,i}(f-L_m^{\Delta',i}f)(x,y)-(f-L_m^{\Delta',i}f)(x,y)]\\ &+\sum_{i=0}^{N}A_i(x,y)(f-L_m^{\Delta',i}f)(x,y)\\ =&\left[f(x,y)\sum_{i=0}^{N}A_i(x,y)-\sum_{i=0}^{N}A_i(x,y)(L_m^{\Delta,i}f)(x,y)\right]\\ &-\sum_{i=0}^{N}A_i(x,y)[(f-L_m^{\Delta',i}f)(x,y)-L_m^{\Delta,i}(f-L_m^{\Delta',i}f)(x,y)]\\ &+\left[f(x,y)\sum_{i=0}^{N}A_i(x,y)-\sum_{i=0}^{N}A_i(x,y)(L_m^{\Delta',i}f)(x,y)\right]\end{aligned}$$

and, finally,

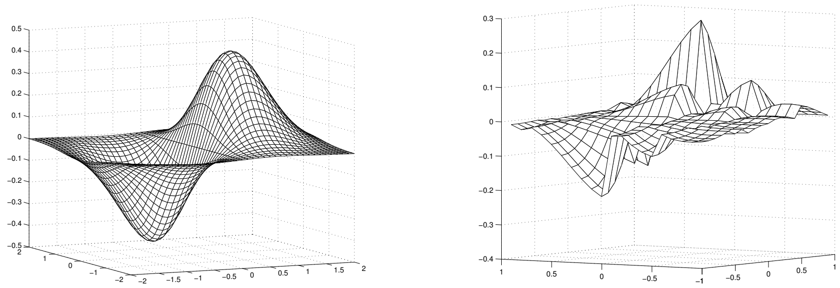
$$(R^{Li}f)(x,y) = \left[f(x,y) - \sum_{i=0}^{N} A_i(x,y)(L_m^{\Delta,i}f)(x,y)\right]$$
$$- \sum_{i=0}^{N} A_i(x,y)[(f - L_m^{\Delta',i}f)(x,y) - L_m^{\Delta,i}(f - L_m^{\Delta',i}f)(x,y)]$$
$$+ \left[f(x,y) - \sum_{i=0}^{N} A_i(x,y)(L_m^{\Delta',i}f)(x,y)\right].$$

Applying Theorem 6 three times the conclusion follows. □

Example 1. Let $f : [-2,2] \times [-2,2] \to \mathbb{R}$,

$$f(x,y) = xe^{-(x^2+y^2)}$$

and consider the nodes $z_1 = (-1,-1)$, $z_2 = (-0.5,-0.5)$, $z_3 = (-0.3,-0.1)$, $z_4 = (0,0)$, $z_5 = (0.5,0.8)$, $z_6 = (1,1)$. In Figure 1 we plot the graphics of f and $S^{Li}f$ for $\mu = 1$. In Figure 2 we plot the error (in absolute value) for Shepard interpolation regarding these data, and also, the error for Shepard interpolation of Lidstone type; we notice that in both cases, the maximum value is 0.5.



(a) Graph of f.

(b) The interpolant $S^{(2)}_{Li}f$, $\mu = 1$.

Fig. 1. Graph of f and $S^{(2)}_{Li}f$.

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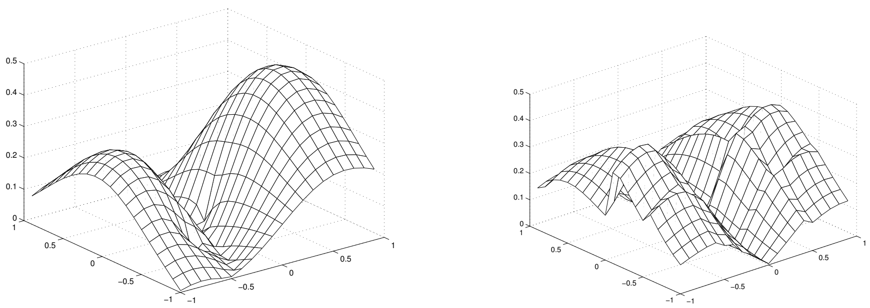

(a) Error for the Shepard interpolation.

(b) Error for Shepard interpolation of Lidstone type.

Fig. 2. Errors.

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