ON THE $L_p$-SATURATION OF THE YE-ZHOU OPERATOR

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Abstract. We solve the saturation problem for a class of Ye-Zhou operator $T_n(f, x) = P_n(x) A_n L_n(f)$ with suitable sequence of matrices $\{A_n\}_{n \geq 1}$. The solution is based on the saturation theorem for the Kantorovich operator established by V. Maier and S. D. Riemenschneider.

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1. INTRODUCTION

For $f \in C[0, 1]$ the $n$th Bernstein polynomial is defined by

$$B_n(f, x) = \sum_{k=0}^{n} p_{n,k}(x) f \left( \frac{k}{n} \right) \equiv \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right).$$

D. X. Zhou showed in [6] for the Kantorovich operator

$$K_n(f, x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) \, dt$$

that if $0 < \alpha < 1$ then $\omega_2(f, t) = \mathcal{O}(t^\alpha)$ if and only if

$$|K_n(f, x) - f(x)| \leq M(x(1-x)/n + 1/n^2)^{\alpha/2}.$$

He also showed [6] that if $1 < \alpha < 2$ then there exist no functions $\{\varphi_{n,\alpha}(x)\}_{n \geq 1}$ such that

$$\omega_2(f, t) = \mathcal{O}(t^\alpha) \Leftrightarrow |K_n(f, x) - f(x)| \leq M \varphi_{n,\alpha}(x).$$

Thus we cannot characterize the second orders of Lipschitz functions by the rate of convergence for the Kantorovich operators. To overcome this difficulty, M. D. Ye and D. X. Zhou [5] introduced a new method of linear approximation by means of matrices: let $P_n(x) \equiv (p_{n,0}(x), \ldots, p_{n,n}(x))$ and

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$L_n(f) = (L_{n,0}(f), \ldots, L_{n,n}(f))^T$, where $\{p_{n,k}(x)\}_{k=0}^n$ is a bases of the linear space spaned $\{1, x, \ldots, x^n\}$ and

$$L_{n,k}(f) = (n+1)^\int_{(k+1)/(n+1)}^{(k+1)/(n+1)} f(t) \, dt$$

are functionals on $C[0,1]$, respectively. Then, for any $(n+1) \times (n+1)$ matrix $A$ we get the so-called Ye-Zhou operator defined by

$$T_n(f,x) = P_n(x) \cdot A \cdot L_n(f).$$

The aim of the present paper is to solve the saturation problem for the class of operator $\{T_n(f,x)\}_{n \geq 1}$ for a suitable sequence of matrices $\{A_n\}_{n \geq 1}$. By reason of [2], the saturation problem of the Ye-Zhou operator will be in connection with the saturation condition of the Kantorovich operator (see [2], [3], [4]).

2. THE CONSTRUCTION OF THE OPERATOR

Let $A_n = (a_{i,j})_{i,j=0}^n$ be an $(n+1) \times (n+1)$ matrix with restriction $a_{i,j} = 0$ for $|i-j| \geq 2$. We denote $a_{i,i} = a_i$, $a_{i,i+1} = b_i$, $a_{i,i-1} = c_i$ and set $c_0 = b_n = 0$. Now, we define the matrix $A_n$ by

$$0 \leq a_i, b_i, c_i \leq 1 \quad \text{for} \quad i = 0, 1, \ldots, n,$$

$$a_i + b_i + c_i = 1 \quad \text{for} \quad i = 0, 1, \ldots, n,$$

$$a_i + b_{i-1} + c_{i+1} = 1 \quad \text{for} \quad i = 0, 1, \ldots, n \quad (b_{-1} = c_{n+1} = 0),$$

$$|ib_i - (i+1)c_i| \leq \frac{1}{2} \quad \text{for} \quad i = 1, 2, \ldots, n-1,$$

$$b_i \leq \frac{M}{n^2} \quad \text{and} \quad c_i \leq \frac{M}{n^2} \quad \text{for} \quad i = 0, 1, \ldots, n,$$

with some absolute constant $M > 0$.

The existence of a matrix $A_n$ with the properties [3]–[7] is guaranteed by the following numerical example:

$$a_0 = 1 - \frac{1}{2n^2}, \quad b_0 = \frac{1}{2n^2},$$

$$a_i = 1 - b_i - c_i, \quad b_i = \frac{i+1}{2n^2}, \quad c_i = \frac{i}{2n^2}, \quad \text{for} \quad i = 1, 2, \ldots, n-1,$$

$$a_n = 1 - \frac{1}{2n^2}, \quad c_n = \frac{1}{2n^2}.$$

Then the explicit expression of the Ye-Zhou operator will be the following:

$$T_n(f,x) = \sum_{i=0}^n p_{n,i}(x)(c_i L_{n,i-1}(f) + a_i L_{n,i}(f) + b_i L_{n,i+1}(f)),$$

where $a_i, b_i, c_i$ ($i = 0, 1, \ldots, n$) satisfy the conditions [3]–[7].

**Theorem 2.1.** Let $A_n$ and $T_n$ be defined as above. Then we have

(i) $T_n$ is a positive operator on $L_p[0,1]$;
(ii) \( T_n(f, x) = \int_0^1 L_n(x, t)f(t) \, dt \), where the kernel

\[
L_n(x, t) = \sum_{i=0}^{n} p_{n,i}(x)(n+1) \left( c_i \chi_{I_{i-1}}(t) + a_i \chi_{I_i}(t) + b_i \chi_{I_{i+1}}(t) \right)
\]

satisfies

\[
\int_0^1 L_n(x, t) \, dx = \int_0^1 L_n(x, t) \, dt = 1
\]

with \( I_i = \left( i/(n+1), (i+1)/(n+1) \right) \) and \( \chi_{I_i} \) the characteristic function on \( I_i \), respectively.

**Proof.** (i) It is a direct consequence of (8) and (3);

(ii) a simple calculation shows that

\[
\int_0^1 p_{n,i}(x) \, dx = \frac{1}{n+1} \quad \text{for} \quad i = 0, 1, \ldots, n.
\]

Then, by (5),

\[
\int_0^1 L_n(x, t) \, dx = \sum_{i=0}^{n} \left( c_i \chi_{I_{i-1}}(t) + a_i \chi_{I_i}(t) + b_i \chi_{I_{i+1}}(t) \right) = c_{i+1} + a_i + b_{i-1} = 1
\]

for \( t \in I_i \), \( i = 0, 1, \ldots, n \), and

\[
\int_0^1 L_n(x, t) \, dt = \sum_{i=0}^{n} p_{n,i}(x)(n+1) \left( \frac{c_i}{n+1} + \frac{a_i}{n+1} + \frac{b_i}{n+1} \right)
\]

\[
= \sum_{i=0}^{n} p_{n,i}(x)(c_i + a_i + b_i)
\]

\[
= 1
\]

in view of (4).

**Theorem 2.2.** For each \( f \in L_p[0, 1] \), \( 1 \leq p \leq \infty \), we have

\[
\lim_{n \to \infty} \| T_n(f) - f \|_p = 0.
\]

**Proof.** In view of (8), (4) and (1) we obtain

\[
T_n(f, x) - K_n(f, x) = \sum_{i=0}^{n} p_{n,i}(x) \{ c_i[L_{n,i-1}(f) - L_{n,i}(f)] + b_i[L_{n,i+1}(f) - L_{n,i}(f)] \}.
\]

If \( p = \infty \) then for \( 1 \leq i \leq n \) one has

\[
|L_{n,i}(f) - L_{n,i-1}(f)| \leq (n+1) \int_{i/(n+1)}^{(i+1)/(n+1)} |f(t)| \, dt + (n+1) \int_{(i-1)/(n+1)}^{i/(n+1)} |f(t)| \, dt
\]

\[
\leq 2 \| f \|_{\infty}
\]
and for $0 \leq i \leq n - 1$
\[
|L_{n,i+1}(f) - L_{n,i}(f)| \leq (n + 1) \int_{i/(n+1)}^{(i+1)/(n+1)} |f(t)| \, dt + (n + 1) \int_{i/(n+1)}^{(i+1)/(n+1)} |f(t)| \, dt
\]
\[
\leq 2 \|f\|_{\infty}.
\]
Thus we have for $x \in [0, 1]$
\[
|T_n(f, x) - K_n(f, x)| \leq \sum_{i=0}^n p_{n,i}(x)(c_i + b_i) 2 \|f\|_{\infty} \leq \frac{4M}{n^2} \|f\|_{\infty}
\]
in view of (7). Hence
\[
(9) \quad \|T_n(f) - K_n(f)\|_{\infty} \leq \frac{4M}{n^2} \|f\|_{\infty}.
\]
If $p = 1$ then for $1 \leq i \leq n$ one has
\[
|L_{n,i}(f) - L_{n,i-1}(f)| \leq (n + 1) \int_{i/(n+1)}^{(i+1)/(n+1)} |f(t)| \, dt + (n + 1) \int_{i/(n+1)}^{(i+1)/(n+1)} |f(t)| \, dt
\]
\[
\leq (n + 1)\|f\|_{1}
\]
and for $0 \leq i \leq n - 1$,
\[
|L_{n,i+1}(f) - L_{n,i}(f)| \leq (n + 1) \int_{i/(n+1)}^{(i+2)/(n+1)} |f(t)| \, dt + (n + 1) \int_{i/(n+1)}^{(i+1)/(n+1)} |f(t)| \, dt
\]
\[
\leq (n + 1)\|f\|_{1}.
\]
Again, by (7), we have for $x \in [0, 1]$
\[
|T_n(f, x) - K_n(f, x)| \leq \sum_{i=0}^n p_{n,i}(x)(c_i + b_i)(n + 1)\|f\|_{1} \leq \frac{4M}{n} \|f\|_{1}.
\]
In conclusion
\[
(10) \quad \|T_n(f) - K_n(f)\|_{1} \leq \frac{4M}{n} \|f\|_{1}.
\]
By the interpolation theorem of Riesz-Thorin we get for $1 \leq p \leq \infty$ that
\[
\|T_n(f) - K_n(f)\|_{p} \leq \frac{4M}{n} \|f\|_{p}
\]
in view of (9) and (10). Hence we obtain the assertion of the theorem, because
\[
\lim_{n \to \infty} \|K_n(f) - f\|_{p} = 0, \quad 1 \leq p \leq \infty.
\]
3. THE SATURATION RESULT

The matrix $A_n$ and the corresponding operators $T_n$ are determined in Section 2. We can now state our main result:

**Theorem 3.1.** Let $\{T_n\}_{n \geq 1}$ be defined as above.

(i) For $f \in L_p[0,1]$, $1 \leq p \leq \infty$, we have

$$\|T_n(f) - f\|_p = O(n^{-1})$$

if and only if

$$f(x) = k + \int_a^x \frac{h(u)}{\varphi(u)} \, du \quad \text{a.e. } x \in [0,1],$$

where $a \in (0,1)$, $\varphi(x) = \sqrt{x(1-x)}$, $k$ is a constant and $h(0) = h(1) = 0$. For $1 < p \leq \infty$, $h$ is absolutely continuous with $h' \in L_p[0,1]$; for $p = 1$, $h$ is of bounded variation on $[0,1]$.

(ii) Moreover, if $\|T_n(f) - f\|_p = o(n^{-1})$ then $f$ is a.e. constant.

**Proof.** The proof is based on the ideas of the saturation theorems for Kantorovich polynomials established by Maier [2], [3] and Riemenschneider [4] (see also [1, pp. 315–321]). Therefore we shall prove only the essential steps regarding the operators $T_n$. To be more precise, a Maier-type inequality (see [2, p. 225] or [1, p. 315, Lemma 6.1]) and a property in connection with the bilinear functional (16) will be established (see [1, p. 320, Lemma 6.6]).

**Lemma 3.2.** Let $g_1(x) = \ln x$, $0 < x \leq 1$. Then there exists an absolute constant $C > 0$ such that

$$|T_n(g_1, x) - g_1(x)| \leq \sum_{j=n+1}^{\infty} \frac{(1-x)^j}{j} + C \sum_{k=0}^{n} \left[ \frac{1}{(k+1)^2} + \frac{1}{n} \right] p_{n,k}(x).$$

**Proof.** Let $S_0 = 0$ and $S_k = \sum_{j=1}^{k} 1/j$ ($k = 1, 2, \ldots$). Then

$$\sum_{k=0}^{n} (S_n - S_k) p_{n,k}(x) = \sum_{k=1}^{n} \frac{1}{k} (1-x)^k$$

(see also [2, p. 225]). Because

$$g_1(x) = \ln x = \ln(1-(1-x)) = - \sum_{k=1}^{\infty} \frac{1}{k} (1-x)^k, \quad x \in (0,1],$$

where $a \in (0,1)$, $\varphi(x) = \sqrt{x(1-x)}$, $k$ is a constant and $h(0) = h(1) = 0$. For $1 < p \leq \infty$, $h$ is absolutely continuous with $h' \in L_p[0,1]$; for $p = 1$, $h$ is of bounded variation on $[0,1]$.
and \( b_n = c_0 = 0 \), then, by (8), (4) and (12), we obtain

\[
T_n(g_1, x) - g_1(x) = 
\]

\[
= \sum_{k=0}^{n} p_{n,k}(x) (c_k L_{n,k-1}(g_1) + a_k L_{n,k}(g_1) + L_{n,k+1}(g_1)) 
\]

\[
+ \sum_{k=0}^{n} (S_n - S_k) p_{n,k}(x) + \sum_{j=n+1}^{\infty} \frac{(1-x)^j}{j} 
\]

\[
= \sum_{k=1}^{n} p_{n,k}(x) c_k \{ L_{n,k-1}(g_1) + (S_n - S_k) \} 
\]

\[
+ \sum_{k=0}^{n} p_{n,k}(x) a_k \{ L_{n,k}(g_1) + (S_n - S_k) \} 
\]

\[
+ \sum_{k=0}^{n-1} p_{n,k}(x) b_k \{ L_{n,k+1}(g_1) + (S_n - S_k) \} + \sum_{j=n+1}^{\infty} \frac{(1-x)^j}{j} 
\]

\[
= \sum_{k=1}^{n} p_{n,k}(x) c_k \{ L_{n,k-1}(g_1) + (S_n - S_k) \} + \sum_{k=1}^{n} p_{n,k}(x) c_k (S_{k-1} - S_k) 
\]

\[
+ \sum_{k=0}^{n} p_{n,k}(x) a_k \{ L_{n,k}(g_1) + (S_n - S_k) \} 
\]

\[
+ \sum_{k=0}^{n-1} p_{n,k}(x) b_k \{ L_{n,k+1}(g_1) + (S_n - S_k) \} 
\]

\[
(13) \quad + \sum_{k=0}^{n-1} p_{n,k}(x) b_k (S_{k+1} - S_k) + \sum_{j=n+1}^{\infty} \frac{(1-x)^j}{j}. 
\]

On the other hand, by [2 p. 225], one has

\[
|L_{n,k}(g_1) + (S_n - S_k)| = \left| (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} \ln t \, dt + (S_n - S_k) \right| 
\]

\[
(14) \quad \leq \frac{5}{6k^2} + \frac{1}{n+1}, 
\]

where \( k = 1, 2, \ldots, n - 1 \) and, by (6) and (3),

\[
\left| \sum_{k=1}^{n} p_{n,k}(x) c_k (S_{k-1} - S_k) + \sum_{k=0}^{n-1} p_{n,k}(x) b_k (S_{k+1} - S_k) \right| = 
\]
\[ \sum_{k=1}^{n} -\frac{c_k}{k} p_{n,k}(x) + \sum_{k=0}^{n-1} \frac{b_k}{k+1} p_{n,k}(x) \]
\[ = -\frac{c_n}{n} p_{n,n}(x) + \sum_{k=1}^{n-1} \left( -\frac{c_k}{k} + \frac{b_k}{k+1} \right) p_{n,k}(x) + b_0 p_{n,0}(x) \]
\[ \leq p_{n,0}(x) + \sum_{k=1}^{n-1} \frac{|kb_n-(k+1)c_k|}{k(k+1)} p_{n,k}(x) + \frac{1}{n} p_{n,n}(x) \]
\[ \leq p_{n,0}(x) + \sum_{k=1}^{n-1} \frac{1}{2k(k+1)} p_{n,k}(x) + \frac{1}{n} p_{n,n}(x) \]
\[ \leq \sum_{k=0}^{n-1} \frac{1}{(k+1)^2} p_{n,k}(x) + \frac{1}{n} p_{n,n}(x). \]

(15)

Combining the relations (13), (14), (15) and (3) we arrive at (11). \( \square \)

Here we mention that we can deduce for \( T_n \) similar statements to the lemmas established in [1, pp. 316–318, Lemmas 6.2, 6.4 and 6.5] for \( K_n \) from the application of Theorem 1.

Furthermore, to prove the necessity in Theorem 3.1 (i), we have to employ the bilinear functionals

\[ A_n(f,\psi) = 2n \int_0^1 \left[ T_n(f,x) - f(x) \right] \psi(x) \, dx. \]

Lemma 3.3. For each fixed \( \psi \in C^2[0,1] \) and \( 1 \leq p \leq \infty \), the functionals (16) have bounded norm on \( L_p[0,1] \):

\[ \|A_n(\cdot,\psi)\|_p \leq C_\psi. \]

Moreover, there exists the limit

\[ \lim_{n \to \infty} A_n(f,\psi) = \int_0^1 f(x)(\varphi^2 \psi')'(x) \, dx. \]

Proof. We have

\[ A_n(f,\psi) = 2n \int_0^1 [K_n(f,x) - f(x)] \psi(x) \, dx + 2n \int_0^1 [T_n(f,x) - K_n(f,x)] \psi(x) \, dx. \]
Using [1] p. 320, Lemma 6.6] and (10) we get

\[ |A_n(f, \psi)| \leq 2n \int_0^1 [K_n(f, x) - f(x)] \psi(x) \, dx \]
\[ + 2n \int_0^1 [T_n(f, x) - K_n(f, x)] \psi(x) \, dx \]
\[ \leq C' \|f\|_1 + 8M \|\psi\|_\infty \|f\|_1 \]
\[ \leq C \|f\|_p, \]

with \( C \) depending on \( \psi \). This inequality implies (17).

By [1] p. 320, (6.14) we have

\[ \lim_{n \to \infty} 2n \int_0^1 [K_n(f, x) - f(x)] \psi(x) \, dx = \int_0^1 f(x)(\varphi^2 \psi')'(x) \, dx, \]

where \( f \in L_p[0, 1] \). On the other hand, by (4), (7) and \( c_0 = b_n = 0 \), we obtain

\[ |2n [T_n(f, x) - K_n(f, x)]| \leq 2n \left\{ \sum_{i=1}^n p_{n,i}(x)c_i \left( |L_{n,i-1}(f)| + |L_{n,i}(f)| \right) \right. \]
\[ \left. + \sum_{i=0}^{n-1} p_{n,i}(x)b_i \left( |L_{n,i+1}(f)| + |L_{n,i}(f)| \right) \right\} \]
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\[
\leq \frac{2M}{n} \left\{ \sum_{i=1}^{n} p_n,i(x) (|L_{n,i-1}(f)| + |L_{n,i}(f)|) + \sum_{i=0}^{n-1} p_n,i(x) (|L_{n,i+1}(f)| + |L_{n,i}(f)|) \right\}.
\]

Furthermore, \(|L_{n,i}(f)| \leq \|f\|_\infty\) for \(f \in C^1[0,1]\). Hence, by (20), one has

\[
|2n [T_n(f,x) - K_n(f,x)]| \leq \frac{8M}{n} \sum_{i=0}^{n} p_n,i(x) \cdot \|f\|_\infty = \frac{8M}{n} \|f\|_\infty.
\]

Thus

\[
2n \int_0^1 [T_n(f,x) - K_n(f,x)] \psi(x) \, dx \leq 2n \|T_n(f) - K_n(f)\|_\infty \cdot \|\psi\|_1 \leq \frac{8M}{n} \|f\|_\infty \cdot \|\psi\|_1
\]

which combined with (19) imply

\[
\lim_{n \to \infty} 2n \int_0^1 [T_n(f,x) - f(x)] \psi(x) \, dx = \int_0^1 f(x)(\varphi^2 \psi')'(x) \, dx =: A(f, \psi).
\]

This \(A(f, \psi)\) is a linear functional on \(L_p[0,1]\). By the Banach-Steinhaus theorem (see e.g. [1], p. 29), \(\lim_{n \to \infty} A_n(f, \psi)\) exists for all \(f \in L_p[0,1]\) and is given by the integral \(\int_0^1 f(x)(\varphi^2 \psi')'(x) \, dx\), that is (18). \(\square\)

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