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ON THE L_p-SATURATION OF THE YE-ZHOU OPERATOR

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Abstract. We solve the saturation problem for a class of Ye-Zhou operator $T_n(f,x) = P_n(x)A_nL_n(f)$ with suitable sequence of matrices $\{A_n\}_{n\geq 1}$. The solution is based on the saturation theorem for the Kantorovich operator established by V. Maier and S. D. Riemenschneider.

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1. INTRODUCTION

For $f \in C[0,1]$ the nth Bernstein polynomial is defined by

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\tfrac{k}{n}\right) \equiv \sum_{k=0}^{n} \tbinom{n}{k} x^k (1-x)^{n-k} f\left(\tfrac{k}{n}\right).$$

D. X. Zhou showed in [6] for the Kantorovich operator

$$K_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,\mathrm{d}t \tag{1}$$

that if $0 < \alpha < 1$ then $\omega_2(f,t) = \mathcal{O}(t^\alpha)$ if and only if

$$|K_n(f,x) - f(x)| \leq M(x(1-x)/n + 1/n^2)^{\alpha/2}.$$

He also showed [6] that if $1 < \alpha < 2$ then there exist no functions $\{\varphi_{n,\alpha}(x)\}_{n\geq 1}$ such that

$$\omega_2(f,t) = \mathcal{O}(t^\alpha) \Leftrightarrow |K_n(f,x) - f(x)| \leq M\varphi_{n,\alpha}(x).$$

Thus we cannot characterize the second orders of Lipschitz functions by the rate of convergence for the Kantorovich operators. To overcome this difficulty, M. D. Ye and D. X. Zhou [5] introduced a new method of linear approximation by means of matrices: let $P_n(x) \equiv (p_{n,0}(x), \ldots, p_{n,n}(x))$ and

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$L_n(f) \equiv (L_{n,0}(f),\ldots,L_{n,n}(f))^{\mathrm{T}}$, where $\{p_{n,k}(x)\}_{k=0}^{n}$ is a bases of the linear space span$\{1,x,\ldots,x^n\}$ and

$$L_{n,k}(f)=(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\ \mathrm{d}t$$

are functionals on $C[0,1]$, respectively. Then, for any $(n+1)\times(n+1)$ matrix A we get the so-called Ye-Zhou operator defined by

$$T_n(f,x)=P_n(x)\cdot A\cdot L_n(f). \tag{2}$$

The aim of the present paper is to solve the saturation problem for the class of operator $\{T_n(f,x)\}_{n\geq 1}$ for a suitable sequence of matrices $\{A_n\}_{n\geq 1}$. By reason of (2), the saturation problem of the Ye-Zhou operator will be in connection with the saturation condition of the Kantorovich operator (see [2], [3], [4]).

2. THE CONSTRUCTION OF THE OPERATOR

Let $A_n=(a_{i,j})_{i,j=0}^{n}$ be an $(n+1)\times(n+1)$ matrix with restriction $a_{i,j}=0$ for $|i-j|\geq 2$. We denote $a_{i,i}=a_i$, $a_{i,i+1}=b_i$, $a_{i,i-1}=c_i$ and set $c_0=b_n=0$. Now, we define the matrix A_n by

$$0\leq a_i,b_i,c_i\leq 1 \quad \text{for} \quad i=0,1,\ldots,n, \tag{3}$$

$$a_i+b_i+c_i=1 \quad \text{for} \quad i=0,1,\ldots,n, \tag{4}$$

$$a_i+b_{i-1}+c_{i+1}=1 \quad \text{for} \quad i=0,1,\ldots,n \quad (b_{-1}=c_{n+1}=0), \tag{5}$$

$$|ib_i-(i+1)c_i|\leq \tfrac{1}{2} \quad \text{for} \quad i=1,2,\ldots,n-1, \tag{6}$$

$$b_i\leq \tfrac{M}{n^2} \quad \text{and} \quad c_i\leq \tfrac{M}{n^2} \quad \text{for} \quad i=0,1,\ldots,n, \tag{7}$$

with some absolute constant $M>0$.

The existence of a matrix A_n with the properties (3)–(7) is guaranteed by the following numerical example:

$$\begin{aligned} a_0&=1-\tfrac{1}{2n^3}, \qquad b_0=\tfrac{1}{2n^3},\\ a_i&=1-b_i-c_i, \quad b_i=\tfrac{i+1}{2n^3}, \quad c_i=\tfrac{i}{2n^3}, \quad \text{for} \quad i=1,2,\ldots,n-1,\\ a_n&=1-\tfrac{1}{2n^2}, \qquad c_n=\tfrac{1}{2n^2}. \end{aligned}$$

Then the explicit expression of the Ye-Zhou operator will be the following:

$$T_n(f,x)=\sum_{i=0}^{n} p_{n,i}(x)(c_iL_{n,i-1}(f)+a_iL_{n,i}(f)+b_iL_{n,i+1}(f)), \tag{8}$$

where a_i,b_i,c_i $(i=0,1,\ldots,n)$ satisfy the conditions (3)–(7).

THEOREM 2.1. *Let A_n and T_n be defined as above. Then we have*

(i) *T_n is a positive operator on $L_p[0,1]$;*

(ii) $T_n(f,x) = \int_0^1 L_n(x,t) f(t)\ \mathrm{d}t$, *where the kernel*

$$L_n(x,t) = \sum_{i=0}^{n} p_{n,i}(x)(n+1)\left(c_i\chi_{I_{i-1}}(t) + a_i\chi_{I_i}(t) + b_i\chi_{I_{i+1}}(t)\right)$$

satisfies

$$\int_0^1 L_n(x,t)\ \mathrm{d}x = \int_0^1 L_n(x,t)\ \mathrm{d}t = 1$$

with $I_i = (i/(n+1), (i+1)/(n+1))$ *and* χ_{I_i} *the characteristic function on* I_i, *respectively.*

Proof. (i) it is a direct consequence of (8) and (3);
(ii) a simple calculation shows that

$$\int_0^1 p_{n,i}(x)\ \mathrm{d}x = \tfrac{1}{n+1} \quad \text{for} \quad i = 0, 1, \ldots, n.$$

Then, by (5),

$$\begin{aligned}\int_0^1 L_n(x,t)\ \mathrm{d}x &= \sum_{i=0}^{n} \left(c_i\chi_{I_{i-1}}(t) + a_i\chi_{I_i}(t) + b_i\chi_{I_{i+1}}(t)\right)\\ &= c_{i+1} + a_i + b_{i-1} = 1\end{aligned}$$

for $t \in I_i$, $i = 0, 1, \ldots, n$, and

$$\begin{aligned}\int_0^1 L_n(x,t)\ \mathrm{d}t &= \sum_{i=0}^{n} p_{n,i}(x)(n+1)\left(\tfrac{c_i}{n+1} + \tfrac{a_i}{n+1} + \tfrac{b_i}{n+1}\right)\\ &= \sum_{i=0}^{n} p_{n,i}(x)(c_i + a_i + b_i)\\ &= 1\end{aligned}$$

in view of (4). □

Theorem 2.2. *For each* $f \in L_p[0,1]$, $1 \le p \le \infty$, *we have*

$$\lim_{n\to\infty} \|T_n(f) - f\|_p = 0.$$

Proof. In view of (8), (4) and (1) we obtain

$$\begin{aligned}&T_n(f,x) - K_n(f,x) =\\ &= \sum_{i=0}^{n} p_{n,i}(x)\{c_i[L_{n,i-1}(f) - L_{n,i}(f)] + b_i[L_{n,i+1}(f) - L_{n,i}(f)]\}.\end{aligned}$$

If $p = \infty$ then for $1 \le i \le n$ one has

$$\begin{aligned}&|L_{n,i}(f) - L_{n,i-1}(f)| \le\\ &\le (n+1)\int_{i/(n+1)}^{(i+1)/(n+1)} |f(t)|\ \mathrm{d}t + (n+1)\int_{(i-1)/(n+1)}^{i/(n+1)} |f(t)|\ \mathrm{d}t\\ &\le 2\ \|f\|_\infty\end{aligned}$$

and for $0 \le i \le n-1$

$$\begin{aligned} &|L_{n,i+1}(f) - L_{n,i}(f)| \le \\ &\le \ (n+1)\int_{(i+1)/(n+1)}^{(i+2)/(n+1)} |f(t)|\ \mathrm{d}t + (n+1)\int_{i/(n+1)}^{(i+1)/(n+1)} |f(t)|\ \mathrm{d}t \\ &\le \ 2\ \|f\|_\infty. \end{aligned}$$

Thus we have for $x \in [0,1]$

$$|T_n(f,x) - K_n(f,x)| \le \sum_{i=0}^{n}\ p_{n,i}(x)(c_i + b_i)\ 2\ \|f\|_\infty \le \tfrac{4M}{n^2}\ \|f\|_\infty$$

in view of (7). Hence

$$\|T_n(f) - K_n(f)\|_\infty \le \tfrac{4M}{n^2}\ \|f\|_\infty. \tag{9}$$

If $p = 1$ then for $1 \le i \le n$ one has

$$\begin{aligned} &|L_{n,i}(f) - L_{n,i-1}(f)| \le \\ &\le \ (n+1)\int_{i/(n+1)}^{(i+1)/(n+1)} |f(t)|\ \mathrm{d}t + (n+1)\int_{(i-1)/(n+1)}^{i/(n+1)} |f(t)|\ \mathrm{d}t \\ &\le \ (n+1)\|f\|_1 \end{aligned}$$

and for $0 \le i \le n-1$,

$$\begin{aligned} &|L_{n,i+1}(f) - L_{n,i}(f)| \le \\ &\le \ (n+1)\int_{(i+1)/(n+1)}^{(i+2)/(n+1)} |f(t)|\ \mathrm{d}t + (n+1)\int_{i/(n+1)}^{(i+1)/(n+1)} |f(t)|\ \mathrm{d}t \\ &\le \ (n+1)\|f\|_1. \end{aligned}$$

Again, by (7), we have for $x \in [0,1]$

$$|T_n(f,x) - K_n(f,x)| \le \sum_{i=0}^{n}\ p_{n,i}(x)(c_i + b_i)(n+1)\|f\|_1 \le \tfrac{4M}{n}\ \|f\|_1.$$

In conclusion

$$\|T_n(f) - K_n(f)\|_1 \le \tfrac{4M}{n}\ \|f\|_1. \tag{10}$$

By the interpolation theorem of Riesz-Thorin we get for $1 \le p \le \infty$ that

$$\|T_n(f) - K_n(f)\|_p \le \tfrac{4M}{n}\ \|f\|_p$$

in view of (9) and (10). Hence we obtain the assertion of the theorem, because

$$\lim_{n\to\infty} \|K_n(f) - f\|_p = 0, \qquad 1 \le p \le \infty. \qquad \square$$

3. THE SATURATION RESULT

The matrix A_n and the corresponding operators T_n are determined in Section 2. We can now state our main result:

THEOREM 3.1. *Let* $\{T_n\}_{n\geq 1}$ *be defined as above.*

(i) *For* $f \in L_p[0,1]$, $1 \leq p \leq \infty$, *we have*

$$\|T_n(f) - f\|_p = \mathcal{O}(n^{-1})$$

if and only if

$$f(x) = k + \int_a^x \tfrac{h(u)}{\varphi^2(u)}\, \mathrm{d}u \quad a.e.\ x \in [0,1],$$

where $a \in (0,1)$, $\varphi(x) = \sqrt{x(1-x)}$, *k is a constant and* $h(0) = h(1) = 0$. *For* $1 < p \leq \infty$, *h is absolutely continuous with* $h' \in L_p[0,1]$; *for* $p = 1$, *h is of bounded variation on* $[0,1]$.

(ii) *Moreover, if* $\|T_n(f) - f\|_p = o(n^{-1})$ *then f is a.e. constant.*

Proof. The proof is based on the ideas of the saturation theorems for Kantorovich polynomials established by Maier [2], [3] and Riemenschneider [4] (see also [1, pp. 315–321]). Therefore we shall prove only the essential steps regarding the operators T_n. To be more precise, a Maier-type inequality (see [2, p. 225] or [1, p. 315, Lemma 6.1]) and a property in connection with the bilinear functional (16) will be established (see [1, p. 320, Lemma 6.6]). □

LEMMA 3.2. *Let* $g_1(x) = \ln x$, $0 < x \leq 1$. *Then there exists an absolute constant* $C > 0$ *such that*

$$|T_n(g_1, x) - g_1(x)| \leq \sum_{j=n+1}^{\infty} \tfrac{(1-x)^j}{j} + C \sum_{k=0}^{n} \left[\tfrac{1}{(k+1)^2} + \tfrac{1}{n}\right] p_{n,k}(x). \tag{11}$$

Proof. Let $S_0 = 0$ and $S_k = \sum_{j=1}^{k} 1/j$ $(k = 1, 2, \ldots)$. Then

$$\sum_{k=0}^{n} (S_n - S_k)\, p_{n,k}(x) = \sum_{k=1}^{n} \tfrac{1}{k}(1-x)^k \tag{12}$$

(see also [2, p. 225]). Because

$$g_1(x) = \ln x = \ln(1 - (1-x)) = -\sum_{k=1}^{\infty} \tfrac{1}{k}(1-x)^k, \quad x \in (0,1],$$

and $b_n = c_0 = 0$, then, by (8), (4) and (12), we obtain

$$\begin{aligned}
&T_n(g_1, x) - g_1(x) = \\
&= \sum_{k=0}^{n} p_{n,k}(x)\left(c_k L_{n,k-1}(g_1) + a_k L_{n,k}(g_1) + L_{n,k+1}(g_1)\right) \\
&\quad + \sum_{k=0}^{n}(S_n - S_k)p_{n,k}(x) + \sum_{j=n+1}^{\infty} \tfrac{(1-x)^j}{j} \\
&= \sum_{k=1}^{n} p_{n,k}(x)c_k\{L_{n,k-1}(g_1) + (S_n - S_k)\} \\
&\quad + \sum_{k=0}^{n} p_{n,k}(x)a_k\{L_{n,k}(g_1) + (S_n - S_k)\} \\
&\quad + \sum_{k=0}^{n-1} p_{n,k}(x)b_k\{L_{n,k+1}(g_1) + (S_n - S_k)\} + \sum_{j=n+1}^{\infty} \tfrac{(1-x)^j}{j}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{n} p_{n,k}(x)c_k\{L_{n,k-1}(g_1) + (S_n - S_{k-1})\} + \sum_{k=1}^{n} p_{n,k}(x)c_k(S_{k-1} - S_k) \\
&\quad + \sum_{k=0}^{n} p_{n,k}(x)a_k\{L_{n,k}(g_1) + (S_n - S_k)\} \\
&\quad + \sum_{k=0}^{n-1} p_{n,k}(x)b_k\{L_{n,k+1}(g_1) + (S_n - S_{k+1})\} \\
&\quad + \sum_{k=0}^{n-1} p_{n,k}(x)b_k(S_{k+1} - S_k) + \sum_{j=n+1}^{\infty} \tfrac{(1-x)^j}{j}. \qquad (13)
\end{aligned}$$

On the other hand, by [2, p. 225], one has

$$\begin{aligned}
|L_{n,k}(g_1) + (S_n - S_k)| &= \left|(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} \ln t \, \mathrm{d}t + (S_n - S_k)\right| \\
&\le \tfrac{5}{6k^2} + \tfrac{1}{n+1}, \qquad (14)
\end{aligned}$$

where $k = 1, 2, \ldots, n-1$ and, by (6) and (3),

$$\left|\sum_{k=1}^{n} p_{n,k}(x)c_k(S_{k-1} - S_k) + \sum_{k=0}^{n-1} p_{n,k}(x)b_k(S_{k+1} - S_k)\right| =$$

$$
\begin{aligned}
&= \Big| \sum_{k=1}^{n} -\tfrac{c_k}{k}\ p_{n,k}(x) + \sum_{k=0}^{n-1} \tfrac{b_k}{k+1}\ p_{n,k}(x) \Big| \\
&= \Big| -\tfrac{c_n}{n}\ p_{n,n}(x) + \sum_{k=1}^{n-1} \Big(-\tfrac{c_k}{k} + \tfrac{b_k}{k+1}\Big)\ p_{n,k}(x) + b_0\ p_{n,0}(x) \Big| \\
&\le\ p_{n,0}(x) + \sum_{k=1}^{n-1} \tfrac{|kb_k-(k+1)c_k|}{k(k+1)}\ p_{n,k}(x) + \tfrac{1}{n}\ p_{n,n}(x) \\
&\le\ p_{n,0}(x) + \sum_{k=1}^{n-1} \tfrac{1}{2k(k+1)}\ p_{n,k}(x) + \tfrac{1}{n}\ p_{n,n}(x) \\
&\le\ p_{n,0}(x) + \sum_{k=1}^{n-1} \tfrac{1}{(k+1)^2}\ p_{n,k}(x) + \tfrac{1}{n}\ p_{n,n}(x) \\
&\le\ \sum_{k=0}^{n-1} \tfrac{1}{(k+1)^2}\ p_{n,k}(x) + \tfrac{1}{n}\ p_{n,n}(x). \qquad (15)
\end{aligned}
$$

Combining the relations (13), (14), (15) and (3) we arrive at (11). □

Here we mention that we can deduce for T_n similar statements to the lemmas established in [1, pp. 316–318, Lemmas 6.2, 6.4 and 6.5] for K_n from the application of Theorem 1.

Furthermore, to prove the necessity in Theorem 3.1 (i), we have to employ the bilinear functionals

$$A_n(f,\psi)\ =\ 2n\int_0^1\ [T_n(f,x)-f(x)]\psi(x)\ \mathrm{d}x. \qquad (16)$$

Lemma 3.3. *For each fixed $\psi \in C^2[0,1]$ and $1 \le p \le \infty$, the functionals (16) have bounded norm on $L_p[0,1]$:*

$$\|A_n(\cdot,\psi)\|_p\ \le\ C_\psi. \qquad (17)$$

Moreover, there exists the limit

$$\lim_{n\to\infty}\ A_n(f,\psi)\ =\ \int_0^1 f(x)(\varphi^2\psi')'(x)\ \mathrm{d}x. \qquad (18)$$

Proof. We have

$$A_n(f,\psi) = 2n\int_0^1 [K_n(f,x)-f(x)]\psi(x)\ \mathrm{d}x + 2n\int_0^1 [T_n(f,x)-K_n(f,x)]\psi(x)\mathrm{d}x.$$

Using [1, p. 320, Lemma 6.6] and (10) we get

$$\begin{aligned} |A_n(f,\psi)| &\le \left|2n\int_0^1 [K_n(f,x)-f(x)]\psi(x)\ \mathrm{d}x\right| \\ &\quad + \left|2n\int_0^1 [T_n(f,x)-K_n(f,x)]\psi(x)\ \mathrm{d}x\right| \\ &\le C'\ \|f\|_1 + 8M\ \|\psi\|_\infty\ \|f\|_1 \\ &\le C\ \|f\|_p, \end{aligned}$$

with C depending on ψ. This inequality implies (17).

By [1, p. 320, (6.14)] we have

$$\lim_{n\to\infty} 2n\int_0^1 [K_n(f,x)-f(x)]\psi(x)\ \mathrm{d}x = \int_0^1 f(x)(\varphi^2\psi')'(x)\ \mathrm{d}x, \tag{19}$$

where $f \in L_p[0,1]$. On the other hand, by (4), (7) and $c_0 = b_n = 0$, we obtain

$$\begin{aligned} |\ 2n\ [T_n(f,x)-K_n(f,x)]| \le 2n\Bigg\{ &\sum_{i=1}^{n} p_{n,i}(x)c_i\left(|L_{n,i-1}(f)| + |L_{n,i}(f)|\right) \\ &+ \sum_{i=0}^{n-1} p_{n,i}(x)b_i\left(|L_{n,i+1}(f)| + |L_{n,i}(f)|\right)\Bigg\} \end{aligned}$$

$$\leq \tfrac{2M}{n}\Big\{ \sum_{i=1}^{n} p_{n,i}(x)\left(|L_{n,i-1}(f)| + |L_{n,i}(f)|\right)$$

$$+ \sum_{i=0}^{n-1} p_{n,i}(x)\left(|L_{n,i+1}(f)| + |L_{n,i}(f)|\right) \Big\}. \tag{20}$$

Furthermore, $|L_{n,i}(f)| \leq \|f\|_\infty$ for $f \in C^1[0,1]$. Hence, by (20), one has

$$|\ 2n\ [T_n(f,x) - K_n(f,x)]\ | \leq \tfrac{8M}{n} \sum_{i=0}^{n} p_{n,i}(x) \cdot \|f\|_\infty = \tfrac{8M}{n}\ \|f\|_\infty.$$

Thus

$$\left|\ 2n \int_0^1 [T_n(f,x) - K_n(f,x)]\psi(x)\ \mathrm{d}x\ \right| \leq 2n\ \|T_n(f) - K_n(f)\|_\infty \cdot \|\psi\|_1$$

$$\leq \tfrac{8M}{n}\ \|f\|_\infty \cdot \|\psi\|_1$$

which combined with (19) imply

$$\lim_{n\to\infty}\ 2n \int_0^1 [T_n(f,x) - f(x)]\psi(x)\ \mathrm{d}x = \int_0^1 f(x)(\varphi^2\psi')'(x)\ \mathrm{d}x =: A(f,\psi).$$

This $A(f,\psi)$ is a linear functional on $L_p[0,1]$. By the Banach-Steinhaus theorem (see e.g. [1, p. 29]), $\lim_{n\to\infty} A_n(f,\psi)$ exists for all $f \in L_p[0,1]$ and is given by the integral $\int_0^1 f(x)(\varphi^2\psi')'(x)\ \mathrm{d}x$, that is (18). □

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