EXTENSIONS OF CERTAIN RELATIONS
OF THE CLASSICAL ANALYSIS

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Abstract. In this work we deal with certain integrals which are convergent for a set of values of a parameter but which become divergent for other values. We extend some relations involving such integrals, replacing them by neutralized ones.

Keywords. Neutrix calculus, neutralized integrals.

1. TWO USEFUL NEUTRICES

We consider the functions $t \mapsto t^{-p} \sin st$, $t \mapsto t^{-p} \cos st$ defined for $t \in [0, \infty[$, where $s, p \in [0, \infty[$ and the following well known relations:

\begin{align*}
\int_0^\infty t^{-p} \cos st \, dt &= \frac{\pi s^{p-1}}{2 \Gamma(p) \cos \frac{p \pi}{2}}, \quad 0 < p < 1, \\
\int_0^\infty t^{-p} \sin st \, dt &= \frac{\pi s^{p-1}}{2 \Gamma(p) \sin \frac{p \pi}{2}}, \quad 0 < p < 2.
\end{align*}

The integral of (1) becomes divergent if $p \geq 1$ and the integral of (2) becomes divergent if $p \geq 2$, so that for these values of parameter $p$ the relations (1) and (2) lose their meaning. To deliver these relations from the restrictions concerning the parameter $p$, we replace the integrals by neutralized ones, using the neutrices $N_{0+}$ and $N_{\infty}$ defined as follows.

$N_{0+}$ is the set constituted by all the linear combinations of the functions defined for $\xi \in [0, \infty[$, $\xi \mapsto \xi^{-q}, q > 0$, $\xi \mapsto C + \ln \xi$ and $o_{0+}(1)$, whose coefficients are arbitrary functions of $s \in [0, \infty[$. We have denoted by $o_{0+}(1)$ a function which tends to 0, when $\xi \to 0$, $\xi > 0$ and by $C$, the constant of Euler.

$N_{\infty}$ is the set constituted by all the linear combinations of the functions defined for $\eta \in [0, \infty[$, $\eta \mapsto \eta^{q} \cos s\eta$, $\eta \mapsto \eta^{q} \sin s\eta$, $q \geq 0$, $\eta \mapsto \ln \eta \cos s\eta$, $\eta \mapsto \ln \eta \sin s\eta$ and $o_{\infty}(1)$, whose coefficients are arbitrary functions of $s \in [0, \infty[$. We have denoted by $o_{\infty}(1)$ a function which tends to 0, as $\eta \to \infty$.

Proposition 1. The set $N_{0+}$ considered above is a normal neutrix.
Proof. A function of $N_{0+}$ has the expression

$$\mu(s, \xi) = \sum_{k=1}^{m} \frac{P_k(s)}{\xi^{q_k}} + P_0(s) \left( C + \ln \xi \right) + R(s) o_{0+}(1),$$

where $0 < q_1 < \cdots < q_m$, $P_k(s)$, $k \in \{1, \cdots, m\}$ and $R(s)$ are arbitrary functions of $s \in ]0, \infty[$ and $P_m$ is not identically zero.

The set $N_{0+}$ is obviously an additive group. To verify the neutrix condition, we suppose that $\mu \in N_{0+}$ and for any $\xi \in ]0, \infty[$, the relation

$$\mu(s, \xi) = \gamma(s)$$

holds, where $\gamma(s)$ is independent of $\xi$. We must show that for each $s \in ]0, \infty[$, $\gamma(s) = 0$.

Because $P_m$ is not identically zero, then $s_0 \in ]0, \infty[$ exists so that $P_m(s_0) \neq 0$. The relation (3) implies

$$\lim_{\xi \to 0+} \xi^{q_m} \mu(s_0, \xi) = P_m(s_0)$$

and the supposition (4) implies

$$\lim_{\xi \to 0+} \xi^{q_m} \mu(s_0, \xi) = \lim_{\xi \to 0+} \xi^{q_m} \gamma(s_0) = 0.$$

Relations (5) and (6) imply $P_m(s_0) = 0$, which is a contradiction. Therefore for each $s \in ]0, \infty[$, $P_m(s) = 0$, that is, $P_m \equiv 0$. In the same way we obtain $P_{m-1} \equiv 0, \cdots, P_1 \equiv 0$. It results that for each $s \in ]0, \infty[$,

$$\mu(s, \xi) = P_0(s) \left( C + \ln \xi \right) + R(s) o_{0+}(1).$$

If $P_0$ were not identically zero, then $s_0 \in ]0, \infty[$ would exist so that $P_0(s_0) \neq 0$. In this case, the limit of $\mu(s_0, \xi)$, when $\xi \to 0$, $\xi > 0$ would be not finite and, according to (4), $\gamma(s_0)$ would be not finite, which is a contradiction. Therefore $P_0 \equiv 0$. From (4) it remains $\gamma(s) = R(s) o_{0+}(1)$ and hence $\gamma(s) = 0$.

The neutrix $N_{0+}$ contains all the functions having the limit zero when $\xi \to 0$, $\xi > 0$ and if it contains a function $\mu$, then it contains also the functions $\alpha \mu$, for any number $\alpha$. Therefore $N_{0+}$ is a normal neutrix. \hfill $\square$

**Proposition 2.** The set $N_\infty$ considered above is also a normal neutrix.

Proof. A function of $N_\infty$ has the expression

$$\nu(s, \eta) = \sum_{k=0}^{m} \left( \frac{P_k(s)}{\eta^{q_k}} \cos \eta + \frac{Q_k(s)}{\eta^{q_k}} \sin \eta \right) \eta^{q_k}$$

$$+ \left( \frac{\tilde{P}(s)}{\eta^{q_k}} \cos \eta + \frac{\tilde{Q}(s)}{\eta^{q_k}} \sin \eta \right) \ln \eta + R(s) o_\infty(1)$$

where $0 \leq q_0 < q_1 < \cdots < q_m$ and $P_k(s), Q_k(s), k \in \{0, \cdots, m\}, \tilde{P}(s), \tilde{Q}(s), R(s)$ are arbitrary functions of $s \in ]0, \infty[$.

The set $N_\infty$ is obviously an additive group. To verify the neutrix condition, we suppose that $\nu \in N_\infty$ and for any $\eta \in ]0, \infty[$, relation

$$\nu(s, \eta) = \gamma(s)$$
holds, where \( \gamma(s) \) is independent of \( \eta \). We must show that for each \( s \in [0, \infty[ \), \( \gamma(s) = 0 \).

We suppose that \( \nu(s, \eta) \) has the expression (7), with \( m \geq 1 \), where at least one of the functions \( P_m \) and \( Q_m \), is not identically 0. Relation (7) implies
\[
\lim_{\eta \to \infty} (\eta^{-q_m} \nu(s, \eta)) = \lim_{\eta \to \infty} (P_m(s) \cos \eta + Q_m(s) \sin \eta)
\]
and relation (8) implies
\[
\lim_{\eta \to \infty} (\eta^{-q_m} \nu(s, \eta)) = 0.
\]
From (9) and (10) it results that the limit of the right member of (9) is 0. Let \( s \in [0, \infty[ \) be an arbitrary but fixed number. The sequences \( (\eta'_n)_{n \in \mathbb{N}^*}, \eta'_n = \frac{n \cdot 2\pi}{s} \) and \( (\eta''_n)_{n \in \mathbb{N}^*}, \eta''_n = \frac{\pi + n \cdot 2\pi}{s} \) tend to \( \infty \) and
\[
\begin{align*}
\lim_{n \to \infty} (P_m(s) \cos \eta'_n + Q_m(s) \sin \eta'_n) &= P_m(s), \\
\lim_{n \to \infty} (P_m(s) \cos \eta''_n + Q_m(s) \sin \eta''_n) &= Q_m(s).
\end{align*}
\]
It results that \( P_m(s) = Q_m(s) = 0 \). Because the number \( s \in [0, \infty[ \) was arbitrarily chosen, it results \( P_m = 0 \), \( Q_m = 0 \). In the same way, we obtain \( P_{m-1} = 0, \cdots, P_1 = 0 \) and \( Q_{m-1} = 0, \cdots, Q_1 = 0 \).

Relation (7) becomes
\[
\nu(s, \eta) = (\tilde{P}(s) \cos \eta + \tilde{Q}(s) \sin \eta) \ln \eta \\
+ P_0(s) \cos \eta + Q_0(s) \sin \eta + R(s) o_\infty(1).
\]
Replacing, according to (8), in this relation \( \nu(s, \eta) \) by \( \gamma(s) \) and dividing the obtained relation by \( \ln \eta \) we obtain
\[
\lim_{\eta \to \infty} (\tilde{P}(s) \cos \eta + \tilde{Q}(s) \sin \eta) = 0.
\]
For an arbitrary fixed \( s \) using again the sequences \( (\eta'_n)_{n \in \mathbb{N}^*} \), and \( (\eta''_n)_{n \in \mathbb{N}^*} \) we obtain \( \tilde{P} \equiv 0 \) and \( \tilde{Q} \equiv 0 \).

Relation (7) becomes
\[
\nu(s, \eta) = P_0(s) \cos \eta + Q_0(s) \sin \eta + R(s) o_\infty(1).
\]
Besides the above sequences \( (\eta'_n)_{n \in \mathbb{N}^*} \) and \( (\eta''_n)_{n \in \mathbb{N}^*} \) we consider also the sequences \( (\eta'''_n)_{n \in \mathbb{N}^*}, \eta'''_n = \frac{(2n+1)\pi}{s} \), and \( (\eta^{iv}_n)_{n \in \mathbb{N}^*}, \eta^{iv}_n = \frac{-\pi + n \cdot 2\pi}{s} \) which all tend to \( \infty \). We obtain
\[
\begin{align*}
\lim_{n \to \infty} (P_0(s) \cos \eta'_n + Q_0(s) \sin \eta'_n) &= P_0(s), \\
\lim_{n \to \infty} (P_0(s) \cos \eta''_n + Q_0(s) \sin \eta''_n) &= Q_0(s), \\
\lim_{n \to \infty} (P_0(s) \cos \eta'''_n + Q_0(s) \sin \eta'''_n) &= -P_0(s), \\
\lim_{n \to \infty} (P_0(s) \cos \eta^{iv}_n + Q_0(s) \sin \eta^{iv}_n) &= -Q_0(s).
\end{align*}
\]
According to (8), for any \( s \in ]0, \infty[ \), the function \( \eta \mapsto \nu(s, \eta) \) has the finite limit \( \gamma(s) \), when \( \eta \to \infty \). From the relations (13) and (15) it results \( P_0(s) = 0 \).

From the relations (14) and (16) it results \( Q_0(s) = 0 \). Because \( s \in ]0, \infty[ \) was arbitrarily chosen, it results \( P_0 \equiv 0, Q_0 \equiv 0 \).

For any \( s \in ]0, \infty[ \) and \( \eta \in ]0, \infty[ \) \( \nu(s, \eta) = R(s)\omega_\infty(1) \), therefore \( \gamma(s) = R(s)\omega_\infty(1) \). This equality may be true only if for each \( s \in ]0, \infty[ \), \( \gamma(s) = 0 \).

The neutrix \( N_{\infty} \) contains all the functions having the limit zero when \( \xi \to \infty \) and if it contains a function \( \nu \), then it contains also the functions \( \alpha \nu \), for any number \( \alpha \). Therefore \( N_{\infty} \) is a normal neutrix.  

\[ \square \]

2. Extension of relations (1) and (2)

For any \( r \in ]0, \infty[ \) and \( s \in ]0, \infty[ \) we consider the neutralized integrals

\[ I_r(s) = \int_{N_{\infty}}^{N_{\infty}} t^{-r} \cos s \xi \cos \xi d \xi, \quad J_r(s) = \int_{N_{\infty}}^{N_{\infty}} t^{-r} \sin s \xi \cos \xi d \xi. \]

**Proposition 3.** For any positive real number \( r \), which differs from an odd integer, relation (1) can be prolonged by \( I_r(s) \), i.e.,

\[ I_r(s) = \frac{s^{r-1}}{2 \Gamma(r)} \cos \frac{\pi s}{2}. \]

And for any positive real number \( r \), which differs from an even integer, the relation (2) can be prolonged by \( J_r(s) \), i.e.,

\[ J_r(s) = \frac{s^{r-1}}{2 \Gamma(r)} \sin \frac{\pi s}{2}. \]

**Proof.** First, we consider the integrals \( I_r \) and \( J_r \), where \( r = p + n, \ p \in ]0, 1[ \) and \( n \in \mathbb{N}^* \). To obtain recurrence relations between these neutralized integrals, we will use the neutralized formula of integration by parts:

\[ I_{p+n}(s) = \left[ -\frac{1}{p+n-1} t^{-p-n+1} \cos s \xi \right]_{N_{\infty}}^{N_{\infty}} J_{N_{\infty}}^{N_{\infty}} + \frac{s}{p+n-1} J_{p+n-1}(s), \]

\[ J_{p+n}(s) = \left[ -\frac{1}{p+n-1} t^{-p-n+1} \sin s \xi \right]_{N_{\infty}}^{N_{\infty}} J_{N_{\infty}}^{N_{\infty}} + \frac{s}{p+n-1} I_{p+n-1}(s). \]

Because

\[ N_{\infty} \lim \eta^{-p-n+1} \cos s \eta = 0, \ N_{\infty} \lim \xi^{-p-n+1} \cos s \xi = 0, \]

\[ N_{\infty} \lim \eta^{-p-n+1} \sin s \eta = 0, \ N_{\infty} \lim \xi^{-p-n+1} \sin s \xi = 0, \]

relations (20) and (21) become

\[ I_{p+n}(s) = -\frac{s}{p+n-1} J_{p+n-1}(s), \ J_{p+n}(s) = \frac{s}{p+n-1} I_{p+n-1}(s). \]

Starting from relations (1) and (2) and using (22), we obtain

\[ I_{p+n}(s) = \frac{s^{p+n-1}}{2 \Gamma(p+n) \cos (p+n)\pi/2}, \]

\[ J_{p+n}(s) = \frac{s^{p+n-1}}{2 \Gamma(p+n) \sin (p+n)\pi/2}. \]

These relations can be easily verified by induction.
To find the expressions of $I_{2k}(s)$, $k \in \mathbb{N}^*$, and of $J_{2k+1}(s)$, $k \in \mathbb{N}$, we use the recurrence relations

$$I_{2k}(s) = -\frac{\pi}{2^{k-1}} J_{2k-1}(s), \quad J_{2k+1}(s) = \frac{\pi}{2^{k}} I_{2k}$$

and the relation $J_1(s) = \frac{\pi}{2}$. We obtain

$$I_{2k}(s) = (\frac{-1}{2})^{k} \frac{\pi}{(2k-1)!} s^{2k-1} \quad \text{and} \quad J_{2k+1}(s) = (\frac{-1}{2})^{k} \frac{\pi}{2^{k}} s^{2k} \frac{\Gamma(2k+1)}{\Gamma(2k)}$$

which completes the proof. □

Now we calculate the values of the neutralized integrals $I_r(s)$ and $J_r(s)$ for the values of $r$ excepted in Proposition 3.

**Proposition 4.** For any $n \in \mathbb{N}^*$, the following relations hold

$$I_{2n+1}(s) = (-1)^n \frac{s^{2n}}{(2n)!} \left( \sum_{k=1}^{2n} \frac{1}{\pi} - \ln s \right),$$

$$J_{2n}(s) = (-1)^{n-1} \frac{s^{2n-1}}{(2n-1)!} \left( \sum_{k=1}^{2n-1} \frac{1}{\pi} - \ln s \right).$$

**Proof.** First we calculate the neutralized integral $I_1(s)$.

$$I_1(s) = \int_{N_0^+}^{N_\infty} t^{-1} \cos st \, dt = \int_{N_0^+}^{N_\infty} t^{-1} \cos st \, dt$$

$$= N_0^+ \lim_{\xi \to 0^+} \int_{\xi}^{\infty} t^{-1} \cos st \, dt = N_0^+ \lim_{\xi \to 0^+} \int_{\xi}^{\infty} x^{-1} \cos x \, dx.$$  \hspace{1cm} (25)

To evaluate the last $N_0^+$ limit, we will use the well know property of the function “cosine integral”

$$\text{Ci}(z) = \int_{z}^{\infty} x^{-1} \cos x \, dx, \quad \lim_{z \to 0^+} (\text{Ci}(z) - \ln z) = C.$$  \hspace{1cm} (27)

where $C$ denotes the Euler’s constant. Relations (27) imply

$$\int_{s\xi}^{\infty} x^{-1} \cos x \, dx = -\text{Ci}(s\xi) = -C - \ln \xi - \ln s + o_0(1),$$

$$N_0^+ \lim_{\xi \to 0^+} \int_{s\xi}^{\infty} x^{-1} \cos x \, dx = -\ln s,$$  \hspace{1cm} (28)

Using the neutralized formula of integration by parts for $I_{2n+1}(s)$ and $J_{2n}(s)$, $n \in \mathbb{N}^*$, we obtain the following recurrence relations:

$$I_{2n+1}(s) = \frac{1}{2n} \left( (-1)^n \frac{s^{2n}}{(2n)!} - s J_{2n}(s) \right),$$  \hspace{1cm} (29)

$$J_{2n}(s) = \frac{1}{2n-1} \left( (-1)^{n-1} \frac{s^{2n-1}}{(2n-1)!} - s I_{2n-1}(s) \right).$$  \hspace{1cm} (30)
From (28), (29) and (30) the relations (25) and (26) result.

3. APPLICATIONS

Remark 1. Formulas (18), (19), (25), (26) give us the values of \( I_r(s) \) and \( J_r(s) \) for each \( r \in [0, \infty[ \) and \( s \in [0, \infty[ \). Further, for each \( r \in [0, \infty[ \) and \( a, b \in [0, \infty[ \), we can easily find the values of the following Froullani-type integrals

\[
F_r(a, b) = \int_{N_0^+}^{N \infty} t^{-r} (\cos bt - \cos at) \, dt,
\]

\[
G_r(a, b) = \int_{N_0^+}^{N \infty} t^{-r} (\sin bt - \sin at) \, dt
\]

using the additivity of neutralized integrals. As example,

\[F_r(a, b) = \frac{\pi}{2} \Gamma(r) \cos \frac{r}{2} \left( b^{r-1} - a^{r-1} \right), \quad \text{if } r \neq 2n + 1, \quad n \in \mathbb{N}, \]

\[G_r(a, b) = \frac{\pi}{2} \Gamma(r) \sin \frac{r}{2} \left( b^{r-1} - a^{r-1} \right), \quad \text{if } r \neq 2n, \quad n \in \mathbb{N}^*, \]

\[F_1(a, b) = \ln a - \ln b. \]

Formula (33) is true also if instead of the neutralized integral we have a classical one. Similarly, for \( r = 2 \), the relation

\[F_2(a, b) = \frac{\pi}{2} (a - b)\]

is also true if we use the classical integral. But these relations can’t be obtained using the additivity of classical integrals because the integrals

\[
\int_0^{\infty} t^{-r} \cos st \, dt
\]

are divergent, for \( r = 1 \) and \( r = 2 \). That illustrate one of the advantages of using neutralized integrals.

REFERENCES


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