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EXTENSIONS OF CERTAIN RELATIONS OF THE CLASSICAL ANALYSIS

GHEORGHE HALIC* and EDUARD HALIC*

Abstract. In this work we deal with certain integrals which are convergent for a set of values of a parameter but which become divergent for other values. We extend some relations involving such integrals, replacing them by neutralized ones.

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1. TWO USEFUL NEUTRICES

We consider the functions $t \mapsto t^{-p} \sin st$, $t \mapsto t^{-p} \cos st$ defined for $t \in]0, \infty[$, where s, $p \in]0, \infty[$ and the following well known relations:

$$\int_0^\infty t^{-p} \cos\ st \,\mathrm{d}t = \frac{\pi\, s^{p-1}}{2\,\Gamma(p)\ \cos\, p\frac{\pi}{2}}, \quad 0 < p < 1, \tag{1}$$

$$\int_0^\infty t^{-p} \sin\ st \,\mathrm{d}t = \frac{\pi\, s^{p-1}}{2\,\Gamma(p)\ \sin\, p\frac{\pi}{2}}, \quad 0 < p < 2. \tag{2}$$

The integral of (1) becomes divergent if $p \geq 1$ and the integral of (2) becomes divergent if $p \geq 2$, so that for these values of parameter p the relations (1) and (2) lose their meaning. To deliver these relations from the restrictions concerning the parameter p, we replace the integrals by neutralized ones, using the neutrices N_{0+} and N_∞ defined as follows.

N_{0+} is the set constituted by all the linear combinations of the functions defined for $\xi \in]0, \infty[$, $\xi \mapsto \xi^{-q}$, $q > 0$, $\xi \mapsto C + \ln\ \xi$ and $o_{0+}(1)$, whose coefficients are arbitrary functions of $s \in]0, \infty[$. We have denoted by $o_{0+}(1)$ a function which tends to 0, when $\xi \to 0$, $\xi > 0$ and by C, the constant of Euler.

N_∞ is the set constituted by all the linear combinations of the functions defined for $\eta \in]0, \infty[$, $\eta \mapsto \eta^q \cos\ s\eta$, $\eta \mapsto \eta^q \sin\ s\eta$, $q \geq 0$, $\eta \mapsto \ln\ \eta\ \cos\ s\eta$, $\eta \mapsto \ln\ \eta\ \sin\ s\eta$ and $o_\infty(1)$, whose coefficients are arbitrary functions of $s \in]0, \infty[$. We have denoted by $o_\infty(1)$ a function which tends to 0, as $\eta \to \infty$.

PROPOSITION 1. *The set N_{0+} considered above is a normal neutrix.*

*"Aurel Vlaicu" University of Arad, Department of Mathematics and Computer Science, Romania, e-mail: georg_halik@yahoo.co.uk, ede_halik@yahoo.de.

Proof. A function of N_{0+} has the expression

$$\mu(s,\xi)=\sum_{k=1}^{m}P_k(s)\ \xi^{-q_k}\ +P_0(s)\ (C+\ln\ \xi)+R(s)\ o_{0+}(1), \tag{3}$$

where $0<q_1<\cdots<q_m,\ P_k(s),\ \ k\in\{1,\cdots,m\}$ and $R(s)$ are arbitrary functions of $s\in]0,\infty[$ and P_m is not identically zero.

The set N_{0+} is obviously an additive group. To verify the neutrix condition, we suppose that $\mu\in N_{0+}$ and for any $\xi\in]0,\infty[$, the relation

$$\mu(s,\xi)=\gamma(s) \tag{4}$$

holds, where $\gamma(s)$ is independent of ξ. We must show that for each $s\in]0,\infty[$, $\gamma(s)=0$.

Because P_m is not identically zero, then $s_0\in]0,\infty[$ exists so that $P_m(s_0)\neq 0$. The relation (3) implies

$$\lim_{\xi\to 0+}\xi^{q_m}\ \mu(s_0,\xi)=P_m(s_0) \tag{5}$$

and the supposition (4) implies

$$\lim_{\xi\to 0+}\xi^{q_m}\ \mu(s_0,\xi)=\lim_{\xi\to 0+}\xi^{q_m}\gamma(s_0)=0. \tag{6}$$

Relations (5) and (6) imply $P_m(s_0)=0$, which is a contradiction. Therefore for each $s\in]0,\infty[$, $P_m(s)=0$, that is, $P_m\equiv 0$. In the same way we obtain $P_{m-1}\equiv 0,\cdots,P_1\equiv 0$. It results that for each $s\in]0,\infty[$,

$$\mu(s,\xi)=P_0(s)\ (C+\ln\ \xi)+R(s)o_{0+}(1).$$

If P_0 were not identically zero, then $s_0\in]0,\infty[$ would exist so that $P_0(s_0)\neq 0$. In this case, the limit of $\mu(s_0,\xi)$, when $\xi\to 0,\ \xi>0$ would be not finite and, according to (4), $\gamma(s_0)$ would be not finite, which is a contradiction. Therefore $P_0\equiv 0$. From (4) it remains $\gamma(s)=R(s)o_{0+}(1)$ and hence $\gamma(s)=0$.

The neutrix N_{0+} contains all the functions having the limit zero when $\xi\to 0$, $\xi>0$ and if it contains a function μ, then it contains also the functions $\alpha\mu$, for any number α. Therefore N_{0+} is a normal neutrix. □

PROPOSITION 2. *The set N_∞ considered above is also a normal neutrix.*

Proof. A function of N_∞ has the expression

$$\begin{aligned}\nu(s,\eta)=&\sum_{k=0}^{m}(\ P_k(s)\ \cos\ s\eta+Q_k(s)\ \sin\ s\eta\ )\ \eta^{q_k}\\ &+(\widetilde{P}(s)\ \cos\ s\eta+\widetilde{Q}(s)\ \sin\ s\eta)\ \ln\ \eta+R(s)o_\infty(1)\end{aligned} \tag{7}$$

where $0\leq q_0<q_1<\cdots<q_m$ and $P_k(s)$, $Q_k(s)$, $k\in\{0,\cdots,m\}$, $\widetilde{P}(s)$, $\widetilde{Q}(s)$, $R(s)$ are arbitrary functions of $s\in]0,\infty[$.

The set N_∞ is obviously an additive group. To verify the neutrix condition, we suppose that $\nu\in N_\infty$ and for any $\eta\in]0,\infty[$, relation

$$\nu(s,\eta)=\gamma(s) \tag{8}$$

holds, where $\gamma(s)$ is independent of η. We must show that for each $s \in]0, \infty[$, $\gamma(s) = 0$.

We suppose that $\nu(s,\eta)$ has the expression (7), with $m \geq 1$, where at least one of the functions P_m and Q_m, is not identically 0. Relation (7) implies

$$(9) \qquad \lim_{\eta\to\infty} \left(\eta^{-q_m}\ \nu(s,\eta)\right) = \lim_{\eta\to\infty} \left(P_m(s)\ \cos\ s\eta + Q_m(s)\ \sin\ s\eta\right)$$

and relation (8) implies

$$(10) \qquad \lim_{\eta\to\infty} \left(\eta^{-q_m}\ \nu(s,\eta)\right) = 0.$$

From (9) and (10) it results that the limit of the right member of (9) is 0. Let $s \in]0,\infty[$ be an arbitrary but fixed number. The sequences $(\eta'_n)_{n\in\mathbb{N}^*}$, $\eta'_n = \frac{n\cdot 2\pi}{s}$ and $(\eta''_n)_{n\in\mathbb{N}^*}$, $\eta''_n = \frac{\frac{\pi}{2} + n\cdot 2\pi}{s}$ tend to ∞ and

$$(11) \qquad \lim_{n\to\infty} \left(P_m(s)\ \cos\ s\eta'_n + Q_m(s)\ \sin\ s\eta'_n\right) = P_m(s),$$

$$(12) \qquad \lim_{n\to\infty} \left(P_m(s)\ \cos\ s\eta''_n + Q_m(s)\ \sin\ s\eta''_n\right) = Q_m(s).$$

It results that $P_m(s) = Q_m(s) = 0$. Because the number $s \in]0,\infty[$ was arbitrarily chosen, it results $P_m \equiv 0$, $Q_m \equiv 0$. In the same way, we obtain $P_{m-1} \equiv 0, \cdots, P_1 \equiv 0$ and $Q_{m-1} \equiv 0, \cdots, Q_1 \equiv 0$.

Relation (7) becomes

$$\nu(s,\eta) = (\widetilde{P}(s)\ \cos\ s\eta + \widetilde{Q}(s)\ \sin\ s\eta)\ \ln\ \eta \\ + P_0(s)\ \cos\ s\eta + Q_0(s)\ \sin\ s\eta + R(s)\ o_\infty(1).$$

Replacing, according to (8), in this relation $\nu(s,\eta)$ by $\gamma(s)$ and dividing the obtained relation by $\ln\ \eta$ we obtain

$$\lim_{\eta\to\infty} (\widetilde{P}(s)\ \cos\ s\eta + \widetilde{Q}(s)\ \sin\ s\eta) = 0.$$

For an arbitrary fixed s using again the sequences $(\eta'_n)_{n\in\mathbb{N}^*}$, and $(\eta''_n)_{n\in\mathbb{N}^*}$ we obtain $\tilde{P} \equiv 0$ and $\tilde{Q} \equiv 0$.

Relation (7) becomes

$$\nu(s,\eta) = P_0(s)\ \cos\ s\eta + Q_0(s)\ \sin\ s\eta + R(s)\, o_\infty(1).$$

Besides the above sequences $(\eta'_n)_{n\in\mathbb{N}^*}$ and $(\eta''_n)_{n\in\mathbb{N}^*}$ we consider also the sequences $(\eta'''_n)_{n\in\mathbb{N}^*}$, $\eta'''_n = \frac{(2n+1)\pi}{s}$, and $(\eta^{iv}_n)_{n\in\mathbb{N}^*}$, $\eta^{iv}_n = \frac{-\frac{\pi}{2} + n\cdot 2\pi}{s}$ which all tend to ∞. We obtain

$$(13) \qquad \lim_{n\to\infty} \left(P_0(s)\ \cos\ s\eta'_n + Q_0(s)\ \sin\ s\eta'_n\right) = P_0(s),$$

$$(14) \qquad \lim_{n\to\infty} \left(P_0(s)\ \cos\ s\eta''_n + Q_0(s)\ \sin\ s\eta''_n\right) = Q_0(s),$$

$$(15) \qquad \lim_{n\to\infty} \left(P_0(s)\ \cos\ s\eta'''_n + Q_0(s)\ \sin\ s\eta'''_n\right) = -\,P_0(s),$$

$$(16) \qquad \lim_{n\to\infty} \left(P_0(s)\ \cos\ s\eta^{iv}_n + Q_0(s)\ \sin\ s\eta^{iv}_n\right) = -\,Q_0(s).$$

According to (8), for any $s \in]0, \infty[$, the function $\eta \mapsto \nu(s, \eta)$ has the finite limit $\gamma(s)$, when $\eta \to \infty$. From the relations (13) and (15) it results $P_0(s) = 0$. From the relations (14) and (16) it results $Q_0(s) = 0$. Because $s \in]0, \infty[$ was arbitrarily chosen, it results $P_0 \equiv 0$, $Q_0 \equiv 0$.

For any $s \in]0, \infty[$ and $\eta \in]0, \infty[$ $\nu(s, \eta) = R(s) o_\infty(1)$, therefore $\gamma(s) = R(s)\, o_\infty(1)$. This equality may be true only if for each $s \in]0, \infty[$, $\gamma(s) = 0$.

The neutrix N_∞ contains all the functions having the limit zero when $\xi \to \infty$ and if it contains a function ν, then it contains also the functions $\alpha\nu$, for any number α. Therefore N_∞ is a normal neutrix. □

2. EXTENSION OF RELATIONS (1) AND (2)

For any $r \in]0, \infty[$ and $s \in]0, \infty[$ we consider the neutralized integrals

$$(17) \qquad I_r(s) = \int_{N_{0+}}^{N_\infty} t^{-r} \cos st \,\mathrm{d}t, \qquad J_r(s) = \int_{N_{0+}}^{N_\infty} t^{-r} \sin st \,\mathrm{d}t.$$

PROPOSITION 3. *For any positive real number r, which differs from an odd integer, relation* (1) *can be prolonged by $I_r(s)$, i.e.,*

$$(18) \qquad I_r(s) = \frac{\pi}{2} \frac{s^{r-1}}{\Gamma(r)\, \cos r \frac{\pi}{2}}$$

and for any positive real number r, which differs from an even integer, the relation (2) *can be prolonged by $J_r(s)$, i.e.,*

$$(19) \qquad J_r(s) = \frac{\pi}{2} \frac{s^{r-1}}{\Gamma(r)\, \sin r \frac{\pi}{2}}.$$

Proof. First, we consider the integrals I_r and J_r, where $r = p + n$, $p \in]0, 1[$ and $n \in \mathbb{N}^*$. To obtain recurrence relations between these neutralized integrals, we will use the neutralized formula of integration by parts:

$$(20) \qquad I_{p+n}(s) = \left[-\frac{1}{p+n-1} t^{-p-n+1} \cos st \right]_{N_{0+}}^{N_\infty} - \frac{s}{p+n-1} J_{p+n-1}(s),$$

$$(21) \qquad J_{p+n}(s) = \left[-\frac{1}{p+n-1} t^{-p-n+1} \sin st \right]_{N_{0+}}^{N_\infty} + \frac{s}{p+n-1} I_{p+n-1}(s).$$

Because

$$N_\infty \lim \eta^{-p-n+1} \cos s\eta = 0, \quad N_{0+} \lim \xi^{-p-n+1} \cos s\xi = 0,$$
$$N_\infty \lim \eta^{-p-n+1} \sin s\eta = 0, \quad N_{0+} \lim \xi^{-p-n+1} \sin s\xi = 0,$$

relations (20) and (21) become

$$(22) \qquad I_{p+n}(s) = -\frac{s}{p+n-1} J_{p+n-1}(s), \quad J_{p+n}(s) = \frac{s}{p+n-1} I_{p+n-1}(s).$$

Starting from relations (1) and (2) and using (22), we obtain

$$I_{p+n}(s) = \frac{\pi}{2} \frac{s^{p+n-1}}{\Gamma(p+n)\, \cos (p+n)\frac{\pi}{2}};$$
$$J_{p+n}(s) = \frac{\pi}{2} \frac{s^{p+n-1}}{\Gamma(p+n)\, \sin (p+n)\frac{\pi}{2}}.$$

These relations can be easily verified by induction.

To find the expressions of $I_{2k}(s),\ k \in \mathbb{N}^*$, and of $J_{2k+1}(s),\ k \in \mathbb{N}$, we use the recurrence relations

$$I_{2k}(s) = -\ \tfrac{s}{2k-1}\ J_{2k-1},\quad J_{2k+1}(s) = \tfrac{s}{2k}\ I_{2k}$$

and the relation $J_1(s) = \frac{\pi}{2}$. We obtain

$$I_{2k}(s) = (-1)^k \tfrac{\pi}{2} \tfrac{s^{2k-1}}{(2k-1)!} = \tfrac{\pi}{2} \tfrac{s^{2k-1}}{\Gamma(2k)\ \cos\ 2k\frac{\pi}{2}}, \tag{23}$$

$$J_{2k+1}(s) = (-1)^k \tfrac{\pi}{2} \tfrac{s^{2k}}{(2k)!} = \tfrac{\pi}{2} \tfrac{s^{2k}}{\Gamma(2k+1)\ \sin\ (2k+1)\frac{\pi}{2}} \tag{24}$$

which completes the proof. □

Now we calculate the values of the neutralized integrals $I_r(s)$ and $J_r(s)$ for the values of r excepted in Proposition 3.

PROPOSITION 4. *For any $n \in \mathbb{N}^*$, the following relations hold*

$$I_{2n+1}(s) = (-1)^n \tfrac{s^{2n}}{(2n)!} \left(\sum_{k=1}^{2n} \tfrac{1}{k}\ -\ \ln\ s \right), \tag{25}$$

$$J_{2n}(s) = (-1)^{n-1} \tfrac{s^{2n-1}}{(2n-1)!} \left(\sum_{k=1}^{2n-1} \tfrac{1}{k}\ -\ \ln\ s \right). \tag{26}$$

Proof. First we calculate the neutralized integral $I_1(s)$.

$$\begin{aligned} I_1(s) &= \int_{N_{0+}}^{N_\infty} t^{-1}\ \cos\ st\ \mathrm{d}t = \int_{N_{0+}}^{N_\infty} t^{-1}\ \cos\ st\ \mathrm{d}t \\ &= N_{0+} \lim \int_\xi^\infty t^{-1}\ \cos\ st\ \mathrm{d}t = N_{0+} \lim \int_{s\xi}^\infty x^{-1}\ \cos\ x\ \mathrm{d}x. \end{aligned}$$

To evaluate the last N_{0+}limit, we will use the well know property of the function "cosine integral"

$$\mathrm{Ci}\,(z) = \int_\infty^z x^{-1}\ \cos\ x\ dx, \qquad \lim_{z \to 0+} (\mathrm{Ci}\,(z) - \ln\ z) = C. \tag{27}$$

where C denotes the Euler's constant. Relations (27) imply

$$\mathrm{Ci}\,(z) = C + \ln\ z + o_{0+}(1),$$

$$\int_{s\xi}^\infty x^{-1}\ \cos\ x\ \mathrm{d}x = -\ \mathrm{Ci}\ (s\xi) = -\,C - \ln\ \xi\ - \ln\ s + o_{0+}(1),$$

$$N_{0+} \lim \int_{s\xi}^\infty x^{-1}\ \cos\ x\ \mathrm{d}x = -\ \ln\ s,$$

$$I_1(s) = -\ \ln\ s \tag{28}$$

Using the neutralized formula of integration by parts for $I_{2n+1}(s)$ and $J_{2n}(s)$, $n \in \mathbb{N}^*$, we obtain the following recurrence relations:

$$I_{2n+1}(s) = \tfrac{1}{2n} \left(\tfrac{(-1)^n s^{2n}}{(2n)!} - s\ J_{2n}(s) \right), \tag{29}$$

$$J_{2n}(s) = \tfrac{1}{2n-1} \left(\tfrac{(-1)^{n-1} s^{2n-1}}{(2n-1)!} - s\ I_{2n-1}(s) \right). \tag{30}$$

From (28), (29) and (30) the relations (25) and (26) result. □

3. APPLICATIONS

REMARK 1. Formulas (18), (19), (25), (26) give us the values of $I_r(s)$ and of $J_r(s)$ for each $r \in]0,\infty[$ and $s \in]0,\infty[$. Further, for each $r \in]0,\infty[$ and $a,\, b \in]0,\infty[$, we can easily find the values of the following Froullani-type integrals

$$F_r(a,b) = \int_{N0+}^{N\infty} t^{-\,r}\ (\cos\ bt\ - \cos\ at)\ \mathrm{d}t,$$

$$G_r(a,b) = \int_{N0+}^{N\infty} t^{-\,r}\ (\sin\ bt\ - \sin\ at)\ \mathrm{d}t$$

using the additivity of neutralized integrals. As example,

(31) $\quad F_r(a,b) = \frac{\pi}{2\ \Gamma(r)\ \cos\ r\frac{\pi}{2}} \left(b^{r-1} - a^{r-1}\right), \quad \text{if} \quad r \neq 2n+1, \quad n \in \mathbb{N},$

(32) $\quad G_r(a.b) = \frac{\pi}{2\ \Gamma(r)\ \sin\ r\ \frac{\pi}{2}} \ (b^{r-1} - \ a^{r-1}), \quad \text{if} \quad r \neq 2n, \quad n \in \mathbb{N}^*,$

(33) $\quad F_1(a,b) = \ln\ a - \ln\ b.$

Formula (33) is true also if instead of the neutralized integral we have a classical one. Similarly, for $r = 2$, the relation

$$F_2(a,b) = \tfrac{\pi}{2}(a-b)$$

is also true if we use the classical integral. But these relations can't be obtained using the additivity of classical integrals because the integrals

$$\int_0^{\infty} t^{-\,r} \cos\ st\, \mathrm{d}t$$

are divergent, for $r = 1$ and $r = 2$. That illustrate one of the advantages of using neutralized integrals. □

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