THE GENERALIZATION OF VORONOVSKAJA’S THEOREM
FOR A CLASS OF LINEAR AND POSITIVE OPERATORS

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Abstract. In this paper we generalize Voronovskaja’s theorem for a class of linear and positive operators, and then, through particular cases, we obtain statements verified by the Bernstein, Schurer, Stancu, Kantorovich and Durrmeyer operators.

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1. INTRODUCTION

In this section, we recall some notions and results which we will use in this article.

Let \( m \) be a nonzero natural number and \( B_m : C([0, 1]) \to C([0, 1]) \) the Bernstein operators, defined for any function \( f \in C([0, 1]) \) by

\[
(B_m f)(x) = \sum_{k=0}^{n} p_{m,k}(x)f\left(\frac{k}{m}\right),
\]

where \( p_{m,k}(x) \) are the fundamental polynomials of Bernstein, defined as follows

\[
p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k},
\]

for any \( x \in [0, 1] \) and any \( k \in \{0, 1, \ldots, m\} \).

In 1932, E. Voronovskaja, proved the result contained in the following theorem.

Theorem 1.1. ([13]) Let \( f \in C([0, 1]) \) be a two times derivable function at the point \( x \in [0, 1] \). Then the equality

\[
\lim_{m \to \infty} m \left[ (B_m f)(x) - f(x) \right] = \frac{x(1-x)}{2} f''(x)
\]

holds.

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For the natural numbers \( m \) and \( p \), \( m \) nonzero, F. Schurer (see [9]) introduced and studied in 1962, the operators \( \bar{B}_{m,p}: C([0,1 + p]) \rightarrow C([0,1]) \), named Bernstein-Schurer operators, defined for any function \( f \in C([0,1 + p]) \) by

\[
(1.4) \quad (\bar{B}_{m,p} f)(x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) f\left(\frac{k}{m}\right),
\]

where \( \tilde{p}_{m,k}(x) \) denotes the fundamental Bernstein-Schurer polynomials, defined as follows

\[
(1.5) \quad \tilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k} = p_{m+p,k}(x)
\]

for any \( x \in [0,1] \) and any \( k \in \{0,1,\ldots,m+p\} \).

In 2002, D. Bărbosu proved the result contained in the following theorem.

**Theorem 1.2.** ([2]) Let \( f \in C([0,1 + p]) \) be a two times derivable function at the point \( x \in [0,1] \). Then the equality

\[
(1.6) \quad \lim_{m \to \infty} (m + p) \left[ (\bar{B}_{m,p} f)(x) - f(x) \right] = px f'(x) + \frac{x(1-x)}{2} f''(x)
\]

holds.

Let \( m \) be a nonzero natural number and the operators \( M_n : L_1([0,1]) \rightarrow C([0,1]) \) are defined for any function \( f \in L_1([0,1]) \) by

\[
(1.7) \quad (M_n f)(x) = (m + 1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) dt,
\]

for any \( x \in [0,1] \).

These operators were introduced in 1967 by J.L. Durrmeyer in [5] and were studied in 1981 by M.M. Derriennic in [4], where the following theorem can be found.

**Theorem 1.3.** Let \( f \in L_1([0,1]) \), bounded on \([0,1]\). If \( f \) is a two times derivable function at the point \( x \in [0,1] \), then

\[
(1.8) \quad \lim_{m \to \infty} m \left[ (M_n f)(x) - f(x) \right] = [x(1-x)f'(x)]'.
\]

If \( f \) is a two times derivable function on \([0,1]\) and the function \( f'' \) is continuous on \([0,1]\), then the convergence from (1.8) is uniform on \([0,1]\).

For \( m \) be a nonzero natural number, let the operators \( K_m : L_1([0,1]) \rightarrow C([0,1]) \) defined for any function \( f \in L_1([0,1]) \) by

\[
(1.9) \quad (K_m f)(x) = (m + 1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t) dt,
\]

for any \( x \in [0,1] \).

The operators \( K_m \), where \( m \) is a nonzero natural number, are named Kantorovich operators, introduced and studied in 1930 by L.V. Kantorovich (see [10]).
For $0 \leq \alpha \leq \beta$ and $m$ a nonzero natural number, define $P_m^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$ for any function $f \in C([0,1])$ by

$$P_m^{(\alpha,\beta)} f (x) = \sum_{k=0}^{m} p_{m,k}(x)f \left( \frac{k+\alpha}{m+\beta} \right),$$

for any $x \in [0,1].$

The operators $P_m^{(\alpha,\beta)}$, where $m$ is a nonzero natural number, are named Bernstein-Stancu operators, introduced and studied in 1969 by D.D. Stancu in the paper [12].

In [12] is the result contained in the following theorem.

**Theorem 1.4.** Let $f \in C([0,1])$ be a two times derivable function at the point $x \in [0,1]$. Then the equality

$$\lim_{m\to\infty} (m + \beta) \left[ (P_m^{(\alpha,\beta)} f) (x) - f (x) \right] = (\alpha - \beta x) f' (x) + \frac{x(1-x)}{2} f'' (x)$$

holds.

We consider $I \subset \mathbb{R}$, $I$ an interval and we shall use the function sets: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on $I$, $B(I) = \{ f | f : I \to \mathbb{R}, \text{ bounded on } I \}$, $C(I) = \{ f | f : I \to \mathbb{R}, \text{ continuous on } I \}$ and $C_B(I) = B(I) \cap C(I)$. For any $x \in I$, let the functions $\varphi_x$, $\psi_x : I \to \mathbb{R}$, $\varphi_x(t) = |t - x|$, $\psi_x(t) = t - x$, for any $t \in I$.

**Definition 1.5.** If $I \subset \mathbb{R}$ is a given interval and $f \in B(I)$, then the first order modulus of smoothness of $f$ is the function $\omega_1 : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega_1(f; \delta) = \sup \{ |f(x') - f(x'')| : x', x' \in I, |x' - x''| \leq \delta \} .$$

In the following, we take into account the properties of the first order modulus of smoothness and the properties of the linear positive functional.

**Lemma 1.6.** If $f \in C_B(I)$, then $\omega_1(f; \cdot)$ have the following properties

a) $\omega_1(f;0) = 0$;

b) $\omega_1(f; \cdot)$ is increasing function;

c) $\omega_1(f; \cdot)$ is uniform continuous function;

for any $\delta > 0$, for any $x,t \in I$, we have

d) $\omega_1(f; \varphi_x(t)) \leq [1 + \delta^{-1} \varphi_x(t)] \omega_1(f; \delta)$

and

e) $|f(t) - f(x)| \leq [1 + \delta^{-2} \psi_x^2(t)] \omega_1(f; \delta)$.

Proof. For proof see [10].

**Lemma 1.7.** Let $A : E(I) \to \mathbb{R}$ be a linear positive functional. Then

a) for any $f,g \in E(I)$ with $f(x) \leq g(x)$, for any $x \in I$, we have $A(f) \leq A(g)$

and

b) $|A(f)| \leq A(|f|)$, for any $f \in E(I)$.

Proof. For proof consult [10].
2. PRELIMINARIES

THEOREM 2.1. Let $I \subset \mathbb{R}$ be an interval, $a \in I$, $n \in \mathbb{N}$ and the function $f : I \rightarrow \mathbb{R}$, $f$ is $n$ times derivable at $a$. According to Taylor’s expansion theorem for the function $f$ around $a$, we have

\begin{equation}
(2.1) \quad f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + (x-a)^n \mu(x-a)
\end{equation}

where $\mu$ is a bounded function and $\lim_{x \to a} \mu(x-a) = 0$.

If $f^{(n)}$ is continuous function on $I$, then for any $\delta > 0$

\begin{equation}
(2.2) \quad |\mu(x-a)| \leq \frac{1}{n!} \left[1 + \delta^{-1}|x-a|\right] \omega_1 \left( f^{(n)}; \delta \right)
\end{equation}

and

\begin{equation}
(2.3) \quad |\mu(x-a)| \leq \frac{1}{n!} \left[1 + \delta^{-2}(x-a)^2\right] \omega_1 \left( f^{(n)}; \delta \right)
\end{equation}

for any $x \in I$.

Proof. If $n = 0$, the proof is immediately. Let $n$ be a nonzero natural number. According to Taylor’s expansion with the Lagrange’s remainder, we have

\begin{equation}
(2.4) \quad f(x) = \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(\xi),
\end{equation}

where $\xi$ is between $a$ and $x$. From (2.1) and (2.4), we obtain $\mu(x-a) = \frac{1}{n!} \left[ f^{(n)}(\xi) - f^{(n)}(a) \right]$ and because $|\xi - a| \leq |x - a|$, we have

\[
|\mu(x-a)| = \frac{1}{n!} \left| f^{(n)}(\xi) - f^{(n)}(a) \right| \\
\leq \frac{1}{n!} \sup_{u,v \in I, |u-v| \leq |x-a|} \left| f^{(n)}(u) - f^{(n)}(v) \right| \\
= \frac{1}{n!} \omega_1 \left( f^{(n)}; |x-a| \right).
\]

Taking Lemma 1.6 into account, the inequalities (2.2) and (2.3) follow. \hfill \square

Let $a, b, a', b' \in \mathbb{R}$ be real numbers, $I \subset \mathbb{R}$ interval, $a < b$, $a' < b'$, $[a, b] \subset I$, $[a', b'] \subset I$ and $[a, b] \cap [a', b'] \neq \emptyset$. For any nonzero natural number $m$, consider the functions $\varphi_{m,k} : I \rightarrow \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in [a', b']$ and any $k \in \{0, 1, \ldots, m\}$ and the linear positive functionals $A_{m,k} : E([a, b]) \rightarrow \mathbb{R}$ for any $k \in \{0, 1, \ldots, m\}$.

DEFINITION 2.2. Let $m$ be a nonzero natural number. Define the operator $L_m : E([a, b]) \rightarrow F(I)$ by

\begin{equation}
(2.5) \quad (L_m f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f),
\end{equation}

for any $f \in E([a, b])$ and for any $x \in I$. 
Proposition 2.3. For $m$ be a nonzero natural number, the $L_m$ operators are linear and positive on $E([a, b] \cap [a', b'])$.

Proof. The proof follows immediately. □

Definition 2.4. Let $m$ be a nonzero natural number and $L_m : E([a, b]) \to F(I)$ be an operator defined in (2.5). For a natural number $i$, define $T_{m,i}^*$

$$
(T_{m,i}^*L_m)(x) = m^i \left( L_m \varphi^i_x \right)(x) = m^i \sum_{k=0}^{m} \varphi_{m,k}(x)A_{m,k}(\psi^i_x),
$$

for any $x \in [a, b] \cap [a', b']$.

3. MAIN RESULTS

In the following, let $s$ be a fixed natural number, $s$ even and we suppose that the operators $(L_m)_{m \geq 1}$ verify the conditions: there exists the smallest $\alpha_s$, $\alpha_s + 2 \in [0, \infty)$ so that

$$
\lim_{m \to \infty} \frac{(T_{m,j}^*L_m)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R}
$$

for any $x \in [a, b] \cap [a', b']$, $j \in \{s, s + 2\}$ and

$$
\alpha_{s+2} < \alpha_s + 2.
$$

Theorem 3.1. Let $f : [a, b] \to \mathbb{R}$ be a function. If $x \in [a, b] \cap [a', b']$ and $f$ is a $s$ times derivable function at $x$, the function $f^{(s)}$ is continuous at $x$, then

$$
\lim_{m \to \infty} m^{s-\alpha_s} \left| (L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i!} \left( T_{m,i}^*L_m \right)(x) \right| = 0.
$$

If $f$ is a $s$ times derivable function on $[a, b]$, the function $f^{(s)}$ is continuous on $[a, b]$ and there exists $m(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ so that for any natural number $m$, $m \geq m(s)$ and for any $x \in [a, b] \cap [a', b']$ we have

$$
\frac{(T_{m,j}^*L_m)(x)}{m^{\alpha_j}} \leq k_j,
$$

where $j \in \{s, s + 2\}$, then the convergence given in (3.3) is uniform on $[a, b] \cap [a', b']$ and

$$
m^{s-\alpha_s} \left| (L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i!} \left( T_{m,i}^*L_m \right)(x) \right| \leq
$$

$$
\leq \frac{1}{m}(k_s + k_{s+2}) \omega_1 \left( f^{(s)}; \frac{1}{\sqrt{m^{2+s+\alpha_s-\alpha_{s+2}}}} \right),
$$

for any $x \in [a, b] \cap [a', b']$, for any natural number $m$, $m \geq m(s)$.
Proof. Let $m$ be a nonzero natural number. According to Taylor’s theorem for the function $f$ around $x$, we have

$$f(t) = \sum_{i=0}^{s} \frac{(t-x)^i}{i!} f^{(i)}(x) + (t-x)^s \mu(t-x),$$

where $\mu$ is a bounded function and $\lim_{t \to x} \mu(t-x) = 0$.

Taking that $A_{m,k}$ is the linear positive functional into account, from (3.6) we have

$$A_{m,k}(f) = \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} A_{m,k}(\psi_x^i) + A_{m,k}(\psi_x^s \mu_x),$$

where $\mu_x : [a, b] \to \mu(t-x)$, for any $t \in [a, b] \cap [a', b']$.

Multiplying by $\varphi_{m,k}(x)$ and summing after $k$, where $k \in \{0, 1, \ldots, m\}$, we obtain

$$(L_m f)(x) = \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} \left( L_m \psi_x^i \right)(x) + \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(\psi_x^s \mu_x),$$

from which

$$m^{s-\alpha_s} \left[ (L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i!} \left( T^s_{m,i} L_m \right)(x) \right] = (R_m f)(x),$$

where

$$(R_m f)(x) = m^{s-\alpha_s} \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(\psi_x^s \mu_x).$$

Then

$$|(R_m f)(x)| \leq m^{s-\alpha_s} \sum_{k=0}^{m} \varphi_{m,k}(x) |A_{m,k}(\psi_x^s \mu_x)|$$

and taking Lemma 1.7 into account, we obtain

$$|(R_m f)(x)| \leq m^{s-\alpha_s} \sum_{k=0}^{m} \varphi_{m,k}(x) |A_{m,k}(\psi_x^s \mu_x)|.$$

According to the relation (2.3), for any $\delta > 0$ and for any $t \in [a, b] \cap [a', b']$, we have

$$|\mu_x(t)| = |\mu(t-x)| \leq \frac{1}{\delta} \left[ 1 + \delta^{-2} \psi_x^2(t) \right] \omega_1(f^{(s)}; \delta),$$

and so

$$(\psi_x^s |\mu_x|)(t) \leq \frac{1}{\delta} \left[ \psi_x^s(t) + \delta^{-2} \psi_x^{s+2}(t) \right] \omega_1(f^{(s)}; \delta).$$
From (3.9) and (3.10), it results that
\[
|(R_m f)(x)| \leq \frac{1}{s!} m^{s-\alpha_s} \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k} (\psi_x^k) + \delta^{-2} \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k} (\psi_x^{k+2}) \right) \omega_1 \left( f(s); \delta \right),
\]
thus
\[
|(R_m f)(x)| \leq \frac{1}{s!} \left[ \left( T^s_{\alpha_s} L_m \right)(x) + \delta^{-2} \left( T^s_{\alpha_s+2} L_m \right)(x) \right] \omega_1 \left( f(s); \delta \right).
\]
Considering \( \delta = \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_s+2}}} \), the inequality above becomes
\[(3.11) \quad |(R_m f)(x)| \leq \frac{1}{s!} \left[ \left( T^s_{\alpha_s} L_m \right)(x) + \frac{\left( T^s_{\alpha_s+2} L_m \right)(x)}{m^{2+\alpha_s+2}} \right] \omega_1 \left( f(s); \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_s+2}}} \right).
\]
Taking (3.1) and (3.2) into account and considering the fact that
\[
\lim_{m \to \infty} \omega_1 \left( f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_s+2}}} \right) = \omega_1 \left( f^{(s)}; 0 \right) = 0,
\]
we have that
\[(3.12) \quad \lim_{m \to \infty} (R_m f)(x) = 0.
\]
From (3.7) and (3.12), (3.3) follows.

If in addition (3.4) takes place, then (3.11) becomes
\[(3.13) \quad |(R_m f)(x)| \leq \frac{1}{s!} \left( k_s + k_{s+2} \right) \omega_1 \left( f(s); \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_s+2}}} \right),
\]
for any natural number \( m, \ m > m(s) \) and for any \( x \in [a, b] \cap [a', b'] \), from which, the convergence from (3.3) is uniform on \([a, b] \cap [a', b'] \). From (3.8) and (3.13), (3.6) follows.

**Corollary 3.2.** Let \( f : [a, b] \to \mathbb{R} \) be a function. If \( x \in [a, b] \cap [a', b'] \) and \( f \) is \( s \) times derivable and the function \( f^{(s)} \) is continuous at \( x \), then
\[(3.14) \quad \lim_{m \to \infty} (L_m f)(x) = f(x) \]
if \( s = 0 \) and
\[(3.15) \quad \lim_{m \to \infty} m^{s-\alpha_s} \left[ (L_m f)(x) - \sum_{i=0}^{s-1} \frac{f^{(i)}(x)}{m^{i+1}} \left( T_{m,i} L_m \right)(x) \right] = \frac{f^{(s)}(x)}{s!} B_s(x)
\]
if \( s \geq 2 \).

If \( f \) is a \( s \) times derivable function on \([a, b] \cap [a', b'] \), the function \( f^{(s)} \) is continuous on \([a, b] \cap [a', b'] \) and (3.4) takes place, then the convergences from (3.14) and (3.15) are uniform on \([a, b] \cap [a', b'] \).

**Proof.** It results from Theorem 3.1.
In the following, consider that \( \varphi_{m,k} = p_{m,k} \) for any \( m, k \) an natural numbers, \( m \neq 0 \) and \( k \in \{0, 1, \ldots, m\} \).

**Application 3.3.** We consider \( a = a' = 0, b = b' = 1 \). For any nonzero natural number \( m \), let the functionals \( A_{m,k} : C([0,1]) \to \mathbb{R}, A_{m,k}(f) = f \left( \frac{k}{m} \right) \), for any \( k \in \{0, 1, \ldots, m\} \) and for any \( f \in C([0,1]) \). In this application, we obtain the Bernstein operators and if \( i \) is a natural number, then

\[
(3.16) \quad \left( T_{m,i}^* B_m \right)(x) = m^i \sum_{k=0}^{m} p_{m,k}(x) \left( \frac{k}{m} - x \right)^i = T_{m,i}(x)
\]

for any \( x \in [0,1] \) (see [6] or [10]).

In [6] are the results contained in the following theorem.

**Theorem 3.4.** If \( i \) is a natural number, then

\[
(3.17) \quad \lim_{m \to \infty} \frac{(T_{m,i}^* B_m)(x)}{m^{\frac{i}{2}}} = [x(1-x)]^{\frac{i}{2}} (a_i x + b_i),
\]

for any \( x \in [0,1] \), where

\[
(3.18) \quad a_i = \begin{cases} 
0, & \text{if } i \text{ is even or } i = 1 \\
-(i-1)!! \sum_{k=1}^{\left[ \frac{i}{2} \right]} \frac{(2k-1)!!}{(2k-2)!!}, & \text{if } i \text{ is odd, } i \geq 3
\end{cases}
\]

and

\[
(3.19) \quad b_i = \begin{cases} 
1, & \text{if } i = 0 \\
0, & \text{if } i = 1 \\
(i-1)!!, & \text{if } i \text{ is even, } i \geq 2 \\
\frac{1}{2} (i-1)!! \sum_{k=1}^{\left[ \frac{i}{2} \right]} \frac{(2k-1)!!}{(2k-2)!!}, & \text{if } i \text{ is odd, } i \geq 3.
\end{cases}
\]

If \( s \) is a natural number, \( s \) even and \( j \in \{s, s + 2\} \), then \( a_j = 0 \),

\[
(3.19) \quad b_j = \begin{cases} 
1, & \text{if } j = 0 \\
(j-1)!!, & \text{if } j \geq 2
\end{cases}
\]

and then from (3.17) it results that there exists a natural number \( m(s) \) so that

\[
(3.20) \quad \left| \frac{(T_{m,j}^* B_m)(x)}{m^{\frac{j}{2}}} - [x(1-x)]^{\frac{j}{2}} b_j \right| < 1
\]

for any \( x \in [0,1] \) and for any natural number \( m, m \geq m(s) \). But \( x(1-x) \leq \frac{1}{4} \) for any \( x \in [0,1] \) and then (3.20) becomes

\[
(3.21) \quad \frac{(T_{m,j}^* B_m)(x)}{m^{\frac{j}{2}}} \leq \left( \frac{1}{4} \right)^{\frac{j}{2}} b_j + 1 = k_j
\]
for any \( x \in [0,1] \) and for any natural number \( m, m \geq m(s) \), where \( j \in \{s, s+2\} \).

Because the conditions (3.2) and (3.4) take place, where \( \alpha_j = \frac{j}{2}, j \in \{s, s+2\} \), Theorem 3.1 and Corollary 3.2 are enounced thus:

**Theorem 3.5.** Let \( f : [0,1] \to \mathbb{R} \) be a function. If \( x \in [0,1] \) and \( f \) is a \( s \) times derivable function at \( x \) and the function \( f^{(s)} \) is continuous at \( x \), then

\[
\lim_{m \to \infty} (B_m f)(x) = f(x)
\]

if \( s = 0 \),

\[
\lim_{m \to \infty} m^\frac{s}{2} \left[ (B_m f)(x) - \sum_{i=0}^{s-1} \frac{f^{(i)}(x)}{m^{i+1}} \left( T_{m,i}^s B_m \right)(x) \right] = 0
\]

if \( s \) is a natural number and

\[
\lim_{m \to \infty} m^\frac{s}{2} \left[ (B_m f)(x) - \sum_{i=0}^{s-1} \frac{f^{(i)}(x)}{m^{i+1}} \left( T_{m,i}^s B_m \right)(x) \right] = \frac{(s-1)!}{s!} [x(1-x)]^\frac{s}{2} f^{(s)}(x)
\]

if \( s \geq 2 \).

If \( f \) is a \( s \) times derivable function on \([0,1]\), the function \( f^{(s)} \) is continuous on \([0,1]\), then the convergences from (3.22)–(3.24) are uniform on \([0,1]\).  

**Remark 3.1.** For \( s = 2 \) in (3.24) we obtain the Voronovskaja’s theorem.

**Application 3.6.** Consider \( a = a' = 0, b = b' = 1 \). For any nonzero natural number \( m \), let the functionals \( A_{m,k} : L_1([0,1]) \to \mathbb{R} \),

\[
A_{m,k}(f) = (m+1) \int_0^1 p_{m,k}(t)f(t)dt,
\]

for any \( k \in \{0,1,\ldots,m\} \) and for any \( f \in L_1([0,1]) \). In this case, we obtain the Durrmeyer operators. With calculus, for \( m \) a nonzero natural number, we have

\[
\left( T_{m,0}^s M_m \right)(x) = 1,
\]

\[
\left( T_{m,1}^s M_m \right)(x) = \frac{m(1-2x)}{m+2},
\]

\[
\left( T_{m,2}^s M_m \right)(x) = m^2 \frac{2(2m-3)x(1-x)+2}{(m+2)(m+3)}
\]

and

\[
\left( T_{m,4}^s M_m \right)(x) = m^4 \frac{12(m^2-21m+10)x(1-x)^2 + 12(6m-10)x(1-x)+24}{(m+2)(m+3)(m+4)}
\]

for any \( x \in [0,1] \).

Then \( \alpha_2 = 1, \alpha_4 = 2 \),

\[
\frac{\left( T_{m,2}^s M_m \right)(x)}{m} < \frac{3}{2},
\]
for any \( m \in \mathbb{N}, m \geq 3 \), any \( x \in [0, 1] \) and

\[
\lim_{m \to \infty} \frac{(T_{m,2}^\ast M_m)(x)}{m} = 2x(1-x).
\]

Then, according to Corollary 3.2, Theorem 1.3 takes place.

If \( m, i \) are natural numbers, let

\[
(T_{m,i}^\ast M_m)(x) = \frac{m!}{i!(s+i)!} \int_0^1 p_{m,k}(t)(x-t)^i \, dt
\]

for any \( x \in [0, 1] \) \( \text{(see [4])} \). Then \( (T_{m,i}^\ast M_m)(x) = m! \left( M_m \psi_i^x \right)(x) \), so

\[
(T_{m,i}^\ast M_m)(x) = (-1)^i m! (m+1) T_{m,i}(x)
\]

for any \( m, i \in \mathbb{N} \) and any \( x \in [0, 1] \).

In [4] the result contained in the following corollary can be found.

**Corollary 3.7.** For any natural number \( j \), \( j \) even, there exists \( k_j \in \mathbb{R} \) so that

\[
m^{\frac{j}{2}+1} T_{m,j}(x) \leq k_j
\]

for any \( m \in \mathbb{N} \), for any \( x \in [0, 1] \) and

\[
\lim_{m \to \infty} m^{\frac{j}{2}+1} T_{m,j}(x) = \frac{j!}{(\frac{j}{2})!} \left[ x(1-x) \right]^\frac{j}{2}.
\]

From (3.29)–(3.31), there exists \( k_j' \in \mathbb{R} \) so that

\[
\frac{(T_{m,i}^\ast M_m)(x)}{m^{\frac{j}{2}}} \leq k_j'
\]

for any \( m \in \mathbb{N} \), for any \( x \in [0, 1] \), where \( j \in \{s, s+2\} \) and

\[
\lim_{m \to \infty} \frac{(T_{m,i}^\ast M_m)(x)}{m^{\frac{s}{2}}} = \frac{s!}{(\frac{s}{2})!} \left[ x(1-x) \right]^\frac{s}{2}.
\]

According to the Corollary 3.2, we have:

**Theorem 3.8.** Let \( f : [0, 1] \to \mathbb{R} \) be a function. If \( x \in [0, 1] \), \( f \) is \( s \) times derivable at \( x \) and the function \( f^{(s)} \) is continuous at \( x \), then

\[
\lim_{m \to \infty} (M_m f)(x) = f(x)
\]

if \( s = 0 \),

\[
\lim_{m \to \infty} m^{\frac{s}{2}} \left[ (M_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^{i+1}} (T_{m,i}^\ast M_m)(x) \right] = 0
\]
if $s$ is a natural number and

$$
(3.36) \quad \lim_{m \to \infty} m^{\frac{s}{2}} \left[ (M_m f)(x) - \sum_{i=0}^{s-1} \frac{f^{(i)}(x)}{m^i} \left( T_{m,i} M_m \right)(x) \right] = \frac{f^{(s)}(x)}{(s!)^2} [x(1-x)]^{s}
$$

if $s \geq 2$.

If $f$ is a $s$ times derivable function on $[0,1]$ and the function $f^{(s)}$ is continuous on $[0,1]$, then the convergence from $(3.34)$–$(3.36)$ are uniform on $[0,1]$.

**Application 3.9.** We consider $a = a' = 0$, $b = b' = 1$. For any nonzero natural number $m$, let the functionals $A_{m,k} : L_1([0,1]) \to \mathbb{R}$,

$$
A_{m,k}(f) = (m+1) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt,
$$

for any $k \in \{0,1,\ldots,m\}$ and for any $f \in L_1([0,1])$. In this case, we obtain the Kantorovich operators. If $i$ is a natural number and $x \in [0,1]$, then

$$
A_{m,k} \left( \psi^i \right) = (m+1) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} (t-x)^i dt
$$

$$
= \frac{m+1}{i+1} \left[ \left( \frac{k+1}{m+1} - x \right)^i - \left( \frac{k}{m+1} - x \right)^i \right]
$$

$$
= \frac{1}{i+1} \sum_{j=0}^{i+1} \left\{ [(k-mx) + (1-x)]^i - [(k-mx) + (-x)]^i \right\},
$$

wherefrom

$$
\left( T_{m,i} K_m \right)(x) = \left( \frac{m}{m+1} \right)^i \frac{1}{i+1} \sum_{k=0}^{m} p_{m,k}(x) \sum_{j=0}^{i+1} \binom{i+1}{j} (k-mx)^j \left[ (1-x)^{i+1-j} - (-x)^{i+1-j} \right],
$$

so

$$
(3.37) \quad \left( T_{m,i} K_m \right)(x) = \left( \frac{m}{m+1} \right)^i \frac{1}{i+1} \sum_{j=0}^{i+1} \binom{i+1}{j} T_{m,j}(x) \left[ (1-x)^{i+1-j} - (-x)^{i+1-j} \right].
$$

Then

$$
\left( T_{m,0} K_m \right)(x) = 1,
$$

$$
\left( T_{m,1} K_m \right)(x) = \frac{m}{2(m+1)} (1-2x),
$$

$$
\left( T_{m,2} K_m \right)(x) = \left( \frac{m}{m+1} \right)^2 \frac{1}{3} (1-x)^3 + x^3 + 3mx(1-x)
$$

and taking into account that $(1-x)^{i+1-j} - (-x)^{i+1-j} < 2$ for any $x \in [0,1]$, $\alpha_2 = 1$ and $\alpha_4 = 2$, we have

$$
(3.38) \quad \frac{\left( T_{m,2} K_m \right)(x)}{m} < 1,
$$
for any $m \in \mathbb{N}$, $m \geq 3$ and for any $x \in [0,1]$. 

According to the Corollary 3.2, we have:

**Theorem 3.10.** Let $f \in L_1([0,1])$, bounded on $[0,1]$. If $f$ is a two times derivable function at the point $x \in [0,1]$ and the function $f''$ is continuous at $x$, then

$$\lim_{m \to \infty} \frac{(T_{m,K_m}(x))}{m^2} < \frac{3}{2},$$

(3.39)

for any $m \in \mathbb{N}$, $m \geq 3$ and for any $x \in [0,1]$.

If $f$ is a two times derivable function on $[0,1]$ and the function $f''$ is continuous on $[0,1]$, then the convergence from (3.40) is uniform on $[0,1]$.

**Application 3.11.** We consider $a = a' = 0$, $b = b' = 1$, $0 \leq \alpha \leq \beta$. For any nonzero natural number $m$, let the functionals $A_{m,k} : [0,1] \to \mathbb{R}$, $A_{m,k}(f) = f(\frac{k+m}{m+1})$, for any $k \in \{0,1,\ldots,m\}$ and for any $f \in C([0,1])$. In this case, we obtain the Stancu operators. With calculus, the conditions in Corollary 3.2 are verified and we have:

**Theorem 3.12.** Let $f \in C([0,1])$. If $f$ is a two times derivable function at the point $x \in [0,1]$ and the function $f''$ is continuous at $x$, then

$$\lim_{m \to \infty} m \left[ (K_m f)(x) - f(x) \right] = \frac{1}{2} [x(1-x)f'(x)]'.$$

(3.40)

If $f$ is a two times derivable function on $[0,1]$ and the function $f''$ is continuous on $[0,1]$, then the convergence from (3.40) is uniform on $[0,1]$.

**Application 3.13.** Let $p$ be a natural number, $m$ be a nonzero natural number, $a = 0$, $b = 1+\frac{1}{p}$, $a' = 0$, $b' = 1$ and the functional $A_{m+p,k}(f) = f \left( \frac{k}{m+p} \right)$, for any $k \in \{0,1,\ldots,m+p\}$ and for any $f \in C([0,1+p])$. In this case, we obtain the Schurer operators (see (1.4) and (1.5)). With calculus, the conditions of Corollary 3.2 are verified and we have:

**Theorem 3.14.** Let $p$ be a natural number and $f \in C([0,1+p])$. If $f$ is a two time derivable function at the point $x \in [0,1]$ and the function $f''$ is continuous at $x$, then

$$\lim_{m \to \infty} (m+p) \left[ (\hat{B}_{m,p} f)(x) - f(x) \right] = px f'(x) + \frac{x(1-x)}{2} f''(x).$$

(3.42)

If $f$ is a two times derivable function on $[0,1]$ and the function $f''$ is continuous on $[0,1]$, then the convergence from (3.42) is uniform on $[0,1]$.

**References**


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