REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 34 (2005) no. 1, pp. 79–91
ictp.acad.ro/jnaat

THE GENERALIZATION OF VORONOVSKAJA'S THEOREM FOR A CLASS OF LINEAR AND POSITIVE OPERATORS

OVIDIU T. POP*

Abstract. In this paper we generalize Voronovskaja's theorem for a class of linear and positive operators, and then, through particular cases, we obtain statements verified by the Bernstein, Schurer, Stancu, Kantorovich and Durrmeyer operators.

MSC 2000. 41A10, 41A36.
Keywords. Bernstein operators, Bernstein-Schurer operators, Bernstein-Stancu operators, Kantorovich operators, Durrmeyer operators, Voronovskaja's theorem.

1. INTRODUCTION

In this section, we recall some notions and results which we will use in this article.

Let m be a nonzero natural number and $B_m : C([0,1]) \to C([0,1])$ the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{n} p_{m,k}(x) f\left(\tfrac{k}{m}\right), \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \tbinom{m}{k} x^k (1-x)^{m-k}, \tag{1.2}$$

for any $x \in [0,1]$ and any $k \in \{0,1,\dots,m\}$.

In 1932, E. Voronovskaja, proved the result contained in the following theorem.

THEOREM 1.1. ([13]) *Let $f \in C([0,1])$ be a two times derivable function at the point $x \in [0,1]$. Then the equality*

$$\lim_{m\to\infty} m\left[(B_m f)(x) - f(x)\right] = \tfrac{x(1-x)}{2}\, f''(x) \tag{1.3}$$

holds.

*National College "Mihai Eminescu", 5 Mihai Eminescu Street, Satu Mare, Romania, e-mail: ovidiutiberiu@yahoo.com.

For the natural numbers m and p, m nonzero, F. Schurer (see [9]) introduced and studied in 1962, the operators $\widetilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$, named Bernstein-Schurer operators, defined for any function $f \in C([0,1+p])$ by

$$(1.4)\qquad (\widetilde{B}_{m,p}f)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\tfrac{k}{m}\right),$$

where $\widetilde{p}_{m,k}(x)$ denotes the fundamental Bernstein-Schurer polynomials, defined as follows

$$(1.5)\qquad \widetilde{p}_{m,k}(x) = \tbinom{m+p}{k} x^k (1-x)^{m+p-k} = p_{m+p,k}(x)$$

for any $x \in [0,1]$ and any $k \in \{0,1,\ldots,m+p\}$.

In 2002, D. Bărbosu proved the result contained in the following theorem.

Theorem 1.2. ([2]) *Let $f \in C([0,1+p])$ be a two times derivable function at the point $x \in [0,1]$. Then the equality*

$$(1.6)\qquad \lim_{m\to\infty} (m+p)\left[(\widetilde{B}_{m,p}f)(x) - f(x)\right] = pxf'(x) + \tfrac{x(1-x)}{2} f''(x)$$

holds.

Let m be a nonzero natural number and the operators $M_n : L_1([0,1]) \to C([0,1])$ are defined for any function $f \in L_1([0,1])$ by

$$(1.7)\qquad (M_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) \mathrm{d}t,$$

for any $x \in [0,1]$.

These operators were introduced in 1967 by J.L. Durrmeyer in [5] and were studied in 1981 by M.M. Derriennic in [4], where the following theorem can be found.

Theorem 1.3. *Let $f \in L_1([0,1])$, bounded on $[0,1]$. If f is a two times derivable function at the point $x \in [0,1]$, then*

$$(1.8)\qquad \lim_{m\to\infty} m\left[(M_m f)(x) - f(x)\right] = \left[x(1-x)f'(x)\right]'.$$

If f is a two times derivable function on $[0,1]$ and the function f'' is continuous on $[0,1]$, then the convergence from (1.8) is uniform on $[0,1]$.

For m be a nonzero natural number, let the operators $K_m : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(1.9)\qquad (K_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t) \mathrm{d}t,$$

for any $x \in [0,1]$.

The operators K_m, where m is a nonzero natural number, are named Kantorovich operators, introduced and studied in 1930 by L.V. Kantorovich (see [10]).

For $0 \leq \alpha \leq \beta$ and m a nonzero natural number, define $P_m^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$ for any function $f \in C([0,1])$ by

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\tfrac{k+\alpha}{m+\beta}\right), \tag{1.10}$$

for any $x \in [0,1]$.

The operators $P_m^{(\alpha,\beta)}$, where m is a nonzero natural number, are named Bernstein-Stancu operators, introduced and studied in 1969 by D.D. Stancu in the paper [12].

In [12] is the result contained in the following theorem.

THEOREM 1.4. *Let $f \in C([0,1])$ be a two times derivable function at the point $x \in [0,1]$. Then the equality*

$$\lim_{m\to\infty} (m+\beta)\left[\left(P_m^{(\alpha,\beta)} f\right)(x) - f(x)\right] = (\alpha - \beta x) f'(x) + \tfrac{x(1-x)}{2} f''(x) \tag{1.11}$$

holds.

We consider $I \subset \mathbb{R}$, I an interval and we shall use the function sets: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on I, $B(I) = \{f | f : I \to \mathbb{R},\ f \text{ bounded on } I\}$, $C(I) = \{f | f : I \to \mathbb{R},\ f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$. For any $x \in I$, let the functions $\varphi_x,\ \psi_x : I \to \mathbb{R}$, $\varphi_x(t) = |t - x|$, $\psi_x(t) = t - x$, for any $t \in I$.

DEFINITION 1.5. If $I \subset \mathbb{R}$ is a given interval and $f \in B(I)$, then the first order modulus of smoothness of f is the function $\omega_1 : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega_1(f;\delta) = \sup\left\{|f(x') - f(x'')| : x',\ x' \in I,\ |x' - x''| \leq \delta\right\}. \tag{1.12}$$

In the following, we take into account the properties of the first order modulus of smoothness and the properties of the linear positive functional.

LEMMA 1.6. *If $f \in C_B(I)$, then $\omega_1(f;\cdot)$ have the following properties*

a) $\omega_1(f;0) = 0$;

b) $\omega_1(f;\cdot)$ is increasing function;

c) $\omega_1(f;\cdot)$ is uniform continuous function;

for any $\delta > 0$, for any $x, t \in I$, we have

d) $\omega_1(f;\varphi_x(t)) \leq \left[1 + \delta^{-1}\varphi_x(t)\right]\omega_1(f;\delta)$

and

e) $|f(t) - f(x)| \leq \left[1 + \delta^{-2}\psi_x^2(t)\right]\omega_1(f;\delta)$.

Proof. For proof see [10]. □

LEMMA 1.7. *Let $A : E(I) \to \mathbb{R}$ be a linear positive functional. Then*

a) for any $f, g \in E(I)$ with $f(x) \leq g(x)$, for any $x \in I$, we have $A(f) \leq A(g)$ and

b) $|A(f)| \leq A(|f|)$, for any $f \in E(I)$.

Proof. For proof consult [10]. □

2. PRELIMINARIES

THEOREM 2.1. *Let $I \subset \mathbb{R}$ be an interval, $a \in I$, $n \in \mathbb{N}$ and the function $f : I \to \mathbb{R}$, f is n times derivable at a. According to Taylor's expansion theorem for the function f around a, we have*

$$f(x) = \sum_{k=0}^{n} \tfrac{(x-a)^k}{k!} f^{(k)}(a) + (x-a)^n \mu(x-a) \tag{2.1}$$

where μ is a bounded function and $\lim\limits_{x\to a} \mu(x-a) = 0$.
If $f^{(n)}$ is continuous function on I, then for any $\delta > 0$

$$|\mu(x-a)| \leq \tfrac{1}{n!} \left[1 + \delta^{-1}|x-a|\right] \omega_1\left(f^{(n)}; \delta\right) \tag{2.2}$$

and

$$|\mu(x-a)| \leq \tfrac{1}{n!} \left[1 + \delta^{-2}(x-a)^2\right] \omega_1\left(f^{(n)}; \delta\right) \tag{2.3}$$

for any $x \in I$.

Proof. If $n = 0$, the proof is immediately. Let n be a nonzero natural number. According to Taylor's expansion with the Lagrange's remainder, we have

$$f(x) = \sum_{k=0}^{n-1} \tfrac{(x-a)^k}{k!} f^{(k)}(a) + \tfrac{(x-a)^n}{n!} f^{(n)}(\xi), \tag{2.4}$$

where ξ is between a and x. From (2.1) and (2.4), we obtain $\mu(x-a) = \frac{1}{n!}\left[f^{(n)}(\xi) - f^{(n)}(a)\right]$ and because $|\xi - a| \leq |x-a|$, we have

$$\begin{aligned} |\mu(x-a)| &= \tfrac{1}{n!}\left|f^{(n)}(\xi) - f^{(n)}(a)\right| \\ &\leq \tfrac{1}{n!} \sup_{\substack{u,v\in I \\ |u-v|\leq|x-a|}} \left|f^{(n)}(u) - f^{(n)}(v)\right| \\ &= \tfrac{1}{n!}\, \omega_1(f^{(n)}; |x-a|). \end{aligned}$$

Taking Lemma 1.6 into account, the inequalities (2.2) and (2.3) follow. □

Let a, b, a', b' be real numbers, $I \subset \mathbb{R}$ interval, $a < b$, $a' < b'$, $[a,b] \subset I$, $[a',b'] \subset I$ and $[a,b] \cap [a',b'] \neq \phi$. For any nonzero natural number m, consider the functions $\varphi_{m,k} : I \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in [a',b']$ and any $k \in \{0, 1, , \ldots, m\}$ and the linear positive functionals $A_{m,k} : E([a,b]) \to \mathbb{R}$ for any $k \in \{0, 1, \ldots, m\}$.

DEFINITION 2.2. *Let m be a nonzero natural number. Define the operator $L_m : E([a,b]) \to F(I)$ by*

$$(L_m f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f), \tag{2.5}$$

for any $f \in E([a,b])$ and for any $x \in$ I.

PROPOSITION 2.3. *For m be a nonzero natural number, the L_m operators are linear and positive on $E([a,b]\cap[a',b'])$.*

Proof. The proof follows immediately. □

DEFINITION 2.4. *Let m be a nonzero natural number and $L_m : E([a,b]) \to F(I)$ be an operator defined in (2.5). For a natural number i, define $T^*_{m,i}$*

$$\left(T^*_{m,i}L_m\right)(x) = m^i\left(L_m\psi^i_x\right)(x) = m^i\sum_{k=0}^{m}\varphi_{m,k}(x)A_{m,k}(\psi^i_x), \tag{2.6}$$

for any $x\in[a,b]\cap[a',b']$.

3. MAIN RESULTS

In the following, let s be a fixed natural number, s even and we suppose that the operators $(L_m)_{m\geq 1}$ verify the conditions: there exists the smallest α_s, $\alpha_{s+2}\in[0,\infty)$ so that

$$\lim_{m\to\infty}\frac{\left(T^*_{m,j}L_m\right)(x)}{m^{\alpha_j}} = B_j\,(x)\in\mathbb{R} \tag{3.1}$$

for any $x\in[a,b]\cap[a',b']$, $j\in\{s,s+2\}$ and

$$\alpha_{s+2}<\alpha_s+2. \tag{3.2}$$

THEOREM 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a function.*

If $x\in[a,b]\cap[a',b']$ and f is a s times derivable function at x, the function $f^{(s)}$ is continuous at x, then

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[(L_mf)(x)-\sum_{i=0}^{s}\frac{f^{(i)}(x)}{m^i i!}\left(T^*_{m,i}L_m\right)(x)\right]=0. \tag{3.3}$$

If f is a s times derivable function on $[a,b]$, the function $f^{(s)}$ is continuous on $[a,b]$ and there exists $m(s)\in\mathbb{N}$ and $k_j\in\mathbb{R}$ so that for any natural number m, $m\geq m(s)$ and for any $x\in[a,b]\cap[a',b']$ we have

$$\frac{\left(T^*_{m,j}L_m\right)(x)}{m^{\alpha_j}}\leq k_j\,, \tag{3.4}$$

where $j\in\{s,s+2\}$, then the convergence given in (3.3) is uniform on $[a,b]\cap[a',b']$ and

$$\begin{aligned} m^{s-\alpha_s}\left|(L_mf)(x)-\sum_{i=0}^{s}\frac{f^{(i)}(x)}{m^i i!}\left(T^*_{m,i}L_m\right)(x)\right| &\leq \\ \leq \frac{1}{s!}\left(k_s+k_{s+2}\right)\omega_1\left(f^{(s)};\frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right)&, \end{aligned} \tag{3.5}$$

for any $x\in[a,b]\cap[a',b']$, for any natural number m , $m\geq m(s)$.

Proof. Let m be a nonzero natural number. According to Taylor's theorem for the function f around x, we have

$$f(t) = \sum_{i=0}^{s} \tfrac{(t-x)^i}{i!} f^{(i)}(x) + (t-x)^s \mu(t-x), \tag{3.6}$$

where μ is a bounded function and $\lim\limits_{t\to x} \mu(t-x) = 0$.

Taking that $A_{m,k}$ is the linear positive functional into account, from (3.6) we have

$$A_{m,k}(f) = \sum_{i=0}^{s} \tfrac{f^{(i)}(x)}{i!} A_{m,k}\left(\psi_x^i\right) + A_{m,k}\left(\psi_x^s \mu_x\right),$$

where $\mu_x : [a,b] \to$, $\mu_x(t) = \mu(t-x)$, for any $t \in [a,b] \cap [a',b']$.

Multiplying by $\varphi_{m,k}(x)$ and summing after k, where $k \in \{0,1,\ldots,m\}$, we obtain

$$(L_m f)(x) = \sum_{i=0}^{s} \tfrac{f^{(i)}(x)}{i!} \left(L_m \psi_x^i\right)(x) + \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}\left(\psi_x^s \mu_x\right),$$

from which

$$m^{s-\alpha_s}\Big[(L_m f)(x) - \sum_{i=0}^{s} \tfrac{f^{(i)}(x)}{m^i i!} \left(T^*_{m,i} L_m\right)(x)\Big] = (R_m f)(x), \tag{3.7}$$

where

$$(R_m f)(x) = m^{s-\alpha_s} \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}\left(\psi_x^s \mu_x\right). \tag{3.8}$$

Then

$$|(R_m f)(x)| \le m^{s-\alpha_s} \sum_{k=0}^{m} \varphi_{m,k}(x) \left|A_{m,k}\left(\psi_x^s \mu_x\right)\right|$$

and taking Lemma 1.7 into account, we obtain

$$|(R_m f)(x)| \le m^{s-\alpha_s} \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}\left(\psi_x^s |\mu_x|\right). \tag{3.9}$$

According to the relation (2.3), for any $\delta > 0$ and for any $t \in [a,b] \cap [a',b']$, we have

$$|\mu_x(t)| = |\mu(t-x)| \le \tfrac{1}{s!}\left[1 + \delta^{-2}\psi_x^2(t)\right]\omega_1(f^{(s)};\delta),$$

and so

$$\left(\psi_x^s |\mu_x|\right)(t) \le \tfrac{1}{s!}\left[\psi_x^s(t) + \delta^{-2}\psi_x^{s+2}(t)\right]\omega_1(f^{(s)};\delta). \tag{3.10}$$

From (3.9) and (3.10), it results that

$$|(R_m f)(x)| \leq \tfrac{1}{s!}\, m^{s-\alpha_s} \left[\sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}\left(\psi_x^s\right) + \right.$$
$$\left. + \delta^{-2} \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}\left(\psi_x^{s+2}\right) \right] \omega_1\left(f^{(s)};\delta\right),$$

thus

$$|(R_m f)(x)| \leq \tfrac{1}{s!}\left[\tfrac{\left(T^*_{m,s} L_m\right)(x)}{m^{\alpha_s}} + \delta^{-2} \tfrac{\left(T^*_{m,s+2} L_m\right)(x)}{m^{\alpha_{s+2}}} m^{-2-\alpha_s+\alpha_{s+2}} \right] \omega_1(f^{(s)};\delta).$$

Considering $\delta = \frac{1}{\sqrt{m^{2+\alpha_2-\alpha_{s+2}}}}$, the inequality above becomes

$$|(R_m f)(x)| \leq \tfrac{1}{s!}\left[\tfrac{\left(T^*_{m,s} L_m\right)(x)}{m^{\alpha_s}} + \tfrac{\left(T^*_{m,s+2} L_m\right)(x)}{m^{\alpha_{s+2}}} \right] \omega_1\Big(f^{(s)}; \tfrac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\Big). \tag{3.11}$$

Taking (3.1) and (3.2) into account and considering the fact that

$$\lim_{m\to\infty} \omega_1\left(f^{(s)}; \tfrac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right) = \omega_1\left(f^{(s)};0\right) = 0,$$

we have that

$$\lim_{m\to\infty} (R_m f)(x) = 0. \tag{3.12}$$

From (3.7) and (3.12), (3.3) follows.

If in addition (3.4) takes place, then (3.11) becomes

$$|(R_m f)(x)| \leq \tfrac{1}{s!}\left(k_s + k_{s+2}\right) \omega_1\left(f^{(s)}; \tfrac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right), \tag{3.13}$$

for any natural number m, $m \geq m(s)$ and for any $x \in [a,b] \cap [a',b']$, from which, the convergence from (3.3) is uniform on $[a,b]\cap[a',b']$. From (3.8) and (3.13), (3.6) follows. □

Corollary 3.2. *Let $f : [a,b] \to \mathbb{R}$ be a function. If $x \in [a,b] \cap [a',b']$ and f is s times derivable and the function $f^{(s)}$ is continuous at x, then*

$$\lim_{m\to\infty} (L_m f)(x) = f(x) \tag{3.14}$$

if $s = 0$ and

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[(L_m f)(x) - \sum_{i=0}^{s-1} \tfrac{f^{(i)}(x)}{m^i i!} \left(T^*_{m,i} L_m\right)(x) \right] = \tfrac{f^{(s)}(x)}{s!}\, B_s(x) \tag{3.15}$$

if $s \geq 2$.

If f is a s times derivable function on $[a,b] \cap [a',b']$, the function $f^{(s)}$ is continuous on $[a,b]\cap[a',b']$ and (3.4) takes place, then the convergences from (3.14) and (3.15) are uniform on $[a,b]\cap[a',b']$.

Proof. It results from Theorem 3.1. □

In the following, consider that $\varphi_{m,k} = p_{m,k}$ for any m, k an natural numbers, $m \neq 0$ and $k \in \{0, 1, \dots, m\}$.

APPLICATION 3.3. *We consider $a = a' = 0$, $b = b' = 1$.For any nonzero natural number m, let the functionals $A_{m,k} : C([0,1]) \to \mathbb{R}$, $A_{m,k}(f) = f\left(\frac{k}{m}\right)$, for any $k \in \{0, 1, \dots, m\}$ and for any $f \in C([0,1])$. In this application, we obtain the Bernstein operators and if i is a natural number, then*

$$\left(T^*_{m,i} B_m\right)(x) = m^i \sum_{k=0}^{m} p_{m,k}(x) \left(\frac{k}{m} - x\right)^i = T_{m,i}(x) \tag{3.16}$$

for any $x \in [0,1]$ (see [6] or [10]).

In [6] are the results contained in the following theorem.

THEOREM 3.4. *If i is a natural number, then*

$$\lim_{m\to\infty} \frac{\left(T^*_{m,i} B_m\right)(x)}{m^{\left[\frac{i}{2}\right]}} = [x(1-x)]^{\left[\frac{i}{2}\right]} (a_i x + b_i), \tag{3.17}$$

for any $x \in [0,1]$, where

$$a_i = \begin{cases} 0, & \text{if } i \text{ is even or } i = 1 \\ -(i-1)!! \sum\limits_{k=1}^{\left[\frac{i}{2}\right]} \frac{(2k-1)!!}{(2k-2)!!}, & \text{if } i \text{ is odd}, \ i \geq 3 \end{cases} \tag{3.18}$$

and

$$b_i = \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{if } i = 1 \\ (i-1)!!, & \text{if } i \text{ is even}, \ i \geq 2 \\ \frac{1}{2}(i-1)!! \sum\limits_{k=1}^{\left[\frac{i}{2}\right]} \frac{(2k-1)!!}{(2k-2)!!}, & \text{if } i \text{ is odd}, \ i \geq 3. \end{cases} \tag{3.19}$$

If s is a natural number, s even and $j \in \{s, s+2\}$, then $a_j = 0$,

$$b_j = \begin{cases} 1, & \text{if } j = 0 \\ (j-1)!!, & \text{if } j \geq 2 \end{cases}$$

and then from (3.17) it results that there exists a natural number $m(s)$ so that

$$\left| \frac{(T^*_{m,j} B_m)(x)}{m^{\frac{j}{2}}} - [x(1-x)]^{\frac{j}{2}}\ b_j \right| < 1 \tag{3.20}$$

for any $x \in [0,1]$ and for any natural number m, $m \geq m(s)$. But $x(1-x) \leq \frac{1}{4}$ for any $x \in [0,1]$ and then (3.20) becomes

$$\frac{(T^*_{m,j} B_m)(x)}{m^{\frac{j}{2}}} \leq \left(\tfrac{1}{4}\right)^{\frac{j}{2}} b_j + 1 = k_j \tag{3.21}$$

for any $x \in [0,1]$ and for any natural number m, $m \geq m(s)$, where $j \in \{s, s+2\}$.

Because the conditions (3.2) and (3.4) take place, where $\alpha_j = \frac{j}{2}$, $j \in \{s, s+2\}$, Theorem 3.1 and Corollary 3.2 are enounced thus:

THEOREM 3.5. *Let $f : [0,1] \to \mathbb{R}$ be a function. If $x \in [0,1]$ and f is a s times derivable function at x and the function $f^{(s)}$ is continuous at x, then*

$$\lim_{m\to\infty} (B_m f)(x) = f(x) \tag{3.22}$$

if $s = 0$,

$$\lim_{m\to\infty} m^{\frac{s}{2}} \Big[(B_m f)(x) - \sum_{i=0}^{s} \tfrac{f^{(i)}(x)}{m^i i!} \left(T^*_{m,i} B_m \right)(x) \Big] = 0 \tag{3.23}$$

if s is a natural number and

$$\lim_{m\to\infty} m^{\frac{s}{2}} \Big[(B_m f)(x) - \sum_{i=0}^{s-1} \tfrac{f^{(i)}(x)}{m^i!} \left(T^*_{m,i} B_m \right)(x) \Big] = \tfrac{(s-1)!!}{s!} \left[x(1-x) \right]^{\frac{s}{2}} f^{(s)}(x) \tag{3.24}$$

if $s \geq 2$.

If f is a s times derivable function on $[0,1]$, the function $f^{(s)}$ is continuous on $[0,1]$, then the convergences from (3.22)–(3.24) are uniform on $[0,1]$.

REMARK 3.1. For $s = 2$ in (3.24) we obtain the Voronovskaja's theorem.

APPLICATION 3.6. *Consider $a = a' = 0$, $b = b' = 1$. For any nonzero natural number m, let the functionals $A_{m,k} : L_1([0,1]) \to \mathbb{R}$,*

$$A_{m,k}(f) = (m+1) \int_0^1 p_{m,k}(t) f(t) \mathrm{d}t,$$

for any $k \in \{0, 1, \ldots, m\}$ and for any $f \in L_1([0,1])$. In this case, we obtain the Durrmeyer operators. With calculus, for m a nonzero natural number, we have

$$\begin{aligned} \left(T^*_{m,0} M_m \right)(x) &= 1, \\ \left(T^*_{m,1} M_m \right)(x) &= \tfrac{m(1-2x)}{m+2}, \\ \left(T^*_{m,2} M_m \right)(x) &= m^2 \tfrac{2(m-3)x(1-x)+2}{(m+2)(m+3)} \end{aligned}$$

and

$$\left(T^*_{m,4} M_m \right)(x) = m^4 \tfrac{12(m^2-21m+10)[x(1-x)]^2 + 12(6m-10)x(1-x) + 24}{(m+2)(m+3)(m+4)(m+5)},$$

for any $x \in [0,1]$.

Then $\alpha_2 = 1$, $\alpha_4 = 2$,

$$\tfrac{\left(T^*_{m,2} M_m \right)(x)}{m} < \tfrac{3}{2}, \tag{3.25}$$

$$\frac{\left(T^*_{m,4}M_m\right)(x)}{m^2} < \frac{7}{4} \tag{3.26}$$

for any $m \in \mathbb{N}$, $m \geq 3$, *any* $x \in [0,1]$ *and*

$$\lim_{m\to\infty} \frac{\left(T^*_{m,2}M_m\right)(x)}{m} = 2x(1-x). \tag{3.27}$$

Then, according to Corollary 3.2, *Theorem* 1.3 *takes place.*

If m, i *are natural numbers, let*

$$\bar{T}_{m,i}(x) = \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t)(x-t)^i \mathrm{d}t \tag{3.28}$$

for any $x \in [0,1]$ (*see* [4]). *Then* $\left(T^*_{m,i}M_m\right)(x) = m^i \left(M_m \psi^i_x\right)(x)$, *so*

$$\left(T^*_{m,i}M_m\right)(x) = (-1)^i m^i (m+1)\bar{T}_{m,i}(x) \tag{3.29}$$

for any $m, i \in \mathbb{N}$ *and any* $x \in [0,1]$.

In [4] the result contained in the following corollary can be found.

Corollary 3.7. *For any natural number* j, j *even, there exists* $k_j \in \mathbb{R}$ *so that*

$$m^{\frac{j}{2}+1}\bar{T}_{m,j}(x) \leq k_j \tag{3.30}$$

for any $m \in \mathbb{N}$, *for any* $x \in [0,1]$ *and*

$$\lim_{m\to\infty} m^{\frac{j}{2}+1}\bar{T}_{m,j}(x) = \frac{j!}{\left(\frac{j}{2}\right)!} \left[x(1-x)\right]^{\frac{j}{2}} . \tag{3.31}$$

From (3.29)–(3.31), there exists $k_j' \in$ so that

$$\frac{(T^*_{m,j}M_m)(x)}{m^{\frac{j}{2}}} \leq k_j' \tag{3.32}$$

for any $m \in$, for any $x \in [0,1]$, where $j \in \{s, s+2\}$ and

$$\lim_{m\to\infty} \frac{(T^*_{m,s}M_m)(x)}{m^{\frac{s}{2}}} = \frac{s!}{\left(\frac{s}{2}\right)!} \left[x(1-x)\right]^{\frac{s}{2}} . \tag{3.33}$$

According to the Corollary 3.2, we have:

Theorem 3.8. *Let* $f : [0,1] \to \mathbb{R}$ *be a function. If* $x \in [0,1]$, f *is* s *times derivable at* x *and the function* $f^{(s)}$ *is continuous at* x, *then*

$$\lim_{m\to\infty} (M_m f)(x) = f(x) \tag{3.34}$$

if $s = 0$,

$$\lim_{m\to\infty} m^{\frac{s}{2}} \left[(M_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} \left(T^*_{m,i}M_m\right)(x)\right] = 0 \tag{3.35}$$

if s is a natural number and

(3.36)

$$\lim_{m\to\infty} m^{\frac{s}{2}}\left[(M_mf)(x)-\sum_{i=0}^{s-1}\frac{f^{(i)}(x)}{m^i i!}\left(T^*_{m,i}M_m\right)(x)\right]=\frac{f^{(s)}(x)}{(\frac{s}{2})!}\,[x(1-x)]^{\frac{s}{2}}$$

if $s\geq 2$.

If f is a s times derivable function on $[0,1]$ and the function $f^{(s)}$ is continuous on $[0,1]$, then the convergence from (3.34)–(3.36) are uniform on $[0,1]$.

Application 3.9. *We consider $a=a'=0$, $b=b'=1$. For any nonzero natural number m, let the functionals $A_{m,k}:L_1([0,1])\to\mathbb{R}$,*

$$A_{m,k}(f)=(m+1)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)\mathrm{d}t,$$

for any $k\in\{0,1,\ldots,m\}$ and for any $f\in L_1([0,1])$. In this case, we obtain the Kantorovich operators. If i is a natural number and $x\in[0,1]$, then

$$\begin{aligned}A_{m,k}\left(\psi_x^i\right)&=(m+1)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}(t-x)^i\mathrm{d}t\\&=\tfrac{m+1}{i+1}\left[\left(\tfrac{k+1}{m+1}-x\right)^{i+1}-\left(\tfrac{k}{m+1}-x\right)^{i+1}\right]\\&=\tfrac{1}{(i+1)(m+1)^i}\left\{[(k-mx)+(1-x)]^{i+1}-[(k-mx)+(-x)^{i+1}\right\},\end{aligned}$$

wherefrom

$$\begin{aligned}&\left(T^*_{m,i}K_m\right)(x)=\\&=\left(\tfrac{m}{m+1}\right)^i\tfrac{1}{i+1}\sum_{k=0}^{m}p_{m,k}(x)\sum_{j=0}^{i+1}\tbinom{i+1}{j}(k-mx)^j\left[(1-x)^{i+1-j}-(-x)^{i+1-j}\right],\end{aligned}$$

so

(3.37)

$$(T^*_{m,i}K_m)(x)=\left(\tfrac{m}{m+1}\right)^i\tfrac{1}{i+1}\sum_{j=0}^{i+1}\tbinom{i+1}{j}T_{m,j}(x)\left[(1-x)^{i+1-j}-(-x)^{i+1-j}\right].$$

Then

$$\begin{aligned}\left(T^*_{m,0}K_m\right)(x)&=1,\\\left(T^*_{m,1}K_m\right)(x)&=\tfrac{m}{2(m+1)}(1-2x),\\\left(T^*_{m,2}K_m\right)(x)&=\left(\tfrac{m}{m+1}\right)^2\tfrac{(1-x)^3+x^3+3mx(1-x)}{3}\end{aligned}$$

and taking into account that $(1-x)^{i+1-j}-(-x)^{i+1-j}<2$ for any $x\in[0,1]$, $\alpha_2=1$ and $\alpha_4=2$, we have

$$\frac{(T^*_{m,2}K_m)(x)}{m}<1, \tag{3.38}$$

$$\frac{(T^*_{m,4}K_m)(x)}{m^2} < \frac{3}{2}\,, \tag{3.39}$$

for any $m \in \mathbb{N}$*,* $m \geq 3$ *and for any* $x \in [0,1]$*.*

According to the Corollary 3.2, we have:

THEOREM 3.10. *Let* $f \in L_1([0,1])$*, bounded on* $[0,1]$*. If* f *is a two times derivable function at the point* $x \in [0,1]$ *and the function* f'' *is continuous at* x*, then*

$$\lim_{m\to\infty} m\left[(K_m f)(x) - f(x)\right] = \frac{1}{2}\left[x(1-x)f'(x)\right]'. \tag{3.40}$$

If f *is a two times derivable function on* $[0,1]$ *and the function* f'' *is continuous on* $[0,1]$*, then the convergence from* (3.40) *is uniform on* $[0,1]$*.*

APPLICATION 3.11. *We consider* $a = a' = 0$*,* $b = b' = 1$*,* $0 \leq \alpha \leq \beta$*. For any nonzero natural number* m*, let the functionals* $A_{m,k} : [0,1] \to \mathbb{R}$*,* $A_{m,k}(f) = f(\frac{k+\alpha}{m+\beta})$*, for any* $k \in \{0,1,\ldots,m\}$ *and for any* $f \in C([0,1])$*. In this case, we obtain the Stancu operators. With calculus, the conditions in Corollary* 3.2 *are verified and we have:*

THEOREM 3.12. *Let* $f \in C([0,1])$*. If* f *is a two times derivable function at the point* $x \in [0,1]$ *and the function* f'' *is continuous at* x*, then*

$$\lim_{m\to\infty} m\left[\left(P_m^{(\alpha,\beta)} f\right)(x) - f(x)\right] = (\alpha - \beta x)f'(x) + \frac{f''(x)}{2}\,x(1-x). \tag{3.41}$$

If f *is a two times derivable function on* $[0,1]$ *and the function* f'' *is continuous on* $[0,1]$*, then the convergence from* (3.41) *is uniform on* $[0,1]$*.*

APPLICATION 3.13. *Let* p *be a natural number,* m *be a nonzero natural number,* $a = 0$*,* $b = 1+p, a' = 0, b' = 1$ *and the functional* $A_{m+p,k}(f) = f\left(\frac{k}{m}\right)$*, for any* $k \in \{0,1,\ldots,m+p\}$ *and for any* $f \in C([0,1+p])$*. In this case, we obtain the Schurer operators* (*see* (1.4) *and* (1.5))*. With calculus, the conditions of Corollary* 3.2 *are verified and we have:*

THEOREM 3.14. *Let* p *be a natural number and* $f \in C\left([0,1+p]\right)$*. If* f *is a two time derivable function at the point* $x \in [0,1]$ *and the function* f'' *is continuous at* x*, then*

$$\lim_{m\to\infty} (m+p)\left[\left(\tilde{B}_{m,p} f\right)(x) - f\left(x\right)\right] = pxf'\left(x\right) + \frac{x(1-x)}{2} f''\left(x\right). \tag{3.42}$$

If f *is a two times derivable function on* $[0,1]$ *and the function* f'' *is continuous on* $[0,1]$*, then the convergence from* (3.42) *is uniform on* $[0,1]$*.*

REFERENCES

[1] AGRATINI, O., *Aproximare prin operatori liniari*, Presa Universitară Clujeană, Cluj-Napoca, 2000.

[2] BĂRBOSU, D., *Voronovskaja Theorem for Bernstein-Schurer operators*, Bul. Şt. Univ. Baia Mare, Ser. B, Matematică-Informatică, **XVIII**, nr. 2, pp. 37–140, 2002.

[3] Bărbosu, D. and Bărbosu, M., *Some properties of the fundamental polynomials of Bernstein-Schurer*, Bul. Şt. Univ. Baia Mare, Ser. B, Matematică-Informatică, **XVIII**, nr. 2, pp. 133–136, 2002.
[4] Derriennic, M. M., *Sur l'approximation de finctions intégrables sur* $[0,1]$ *par des polynômes de Bernstein modifiés*, J. Approx. Theory, **31**, pp. 325–343, 1981.
[5] Durrmeyer, J. L., *Une formule d'inversion de la transformeé de Laplace: Applications à la théorie des moments, Thèse de 3e cycle*, Faculté des Sciences de l'Université de Paris, 1967.
[6] Lorentz, G. G., *Bernstein polynomials*, University of Toronto Press, Toronto, 1953.
[7] Lorentz, G. G., *Aproximation of functions*, Holt, Rinehart and Winston, New York, 1966.
[8] Pop, O. T., *About a class of linear and positive operators* (to appear in Proced. of ICAM4).
[9] Schurer, F., *Linear positive operators in approximation theory*, Math. Inst. Techn., Univ. Delft. Report, 1962.
[10] Stancu, D. D., Coman, Gh., Agratini, O., and Trîmbiţaş, R., *Analiză numerică şi teoria aproximării*, **I**, Presa Universitară Clujeană, Cluj-Napoca, 2001.
[11] Stancu, D. D., *Curs şi culegere de probleme de analiză numerică*, **I**, Univ. "Babeş-Bolyai" Cluj-Napoca, Facultatea de Matematică, Cluj-Napoca, 1977.
[12] Stancu, D. D., *Asupra unei generalizări a polinoamelor lui Bernstein*, Studia Univ. Babeş-Bolyai, Ser. Math.-Phys., **14**, pp. 31–45, 1969.
[13] Voronovskaja, E., *Détermination de la forme asymtotique d'approximation des fonctions par les polynômes de M. Bernstein*, C. R. Acad. Sci. URSS, pp. 79–85, 1932.

Received by the editors: January 17, 2005.