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ON THE ASYMPTOTIC BEHAVIOR OF L_p EXTREMAL POLYNOMIALS

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Abstract. Let β denotes a positive Szegö measure on the unit circle Γ and δ_{z_k}denotes an anatomic measure (δ Dirac) centered on the point z_k. We study, for all $p>0$, the asymptotic behavior of L_p extremal polynomials with respect to a measure of the type

$$\alpha=\beta+\sum_{k=1}^{\infty}A_k\delta_{z_k},$$

where $\{z_k\}_{k=1}^{\infty}$ is an infinite collection of points outside Γ.

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1. INTRODUCTION

Let α be a finite measure defined on the borel sets of C and of compact support F. We denote by $m_{n,p}(\alpha,F)$, $n\in\mathbb{N}$, $p>0$ the extremal constants

$$m_{n,p}(\alpha,F)=\min\left\{\begin{array}{c}\|Q_n\|_{L_p(\alpha,F)}:Q_n=z^n+a_{n-1}z^{n-1}+...+a_0,\\ a_0,...,a_{n-1}\in C\end{array}\right\},$$

and by $T_{n,p}(\alpha,F)$ the associated extremal polynomials (we suppose that $z^n\in L_p(\alpha,F)$, $n\in\mathbb{N}$). The case $p=2$ is the special case of $L_2(\alpha,F)$ monic orthogonal polynomials.

There are many interesting problems about orthogonal or extremal polynomials. The most important and difficult ones are their asymptotic behavior and zero distributions.

The study of the asymptotic behavior of orthogonal or extremal polynomials contributed in the resolution of other important problems in Mathematics. We especially mention:

(i) The convergence of Padé approximants ($F=[-1,+1]\cup\{z_k\}$, $p=2$, see [7])
(ii) The spectral theory ($F=[-1,+1]\cup\{z_k\}$, $p=2$, see [5], [22])

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(iii) The zero distribution of extremal polynomials ($F = \Gamma = \{z : |z| = 1\}$, $p \geq 1$, see [16], $F = \Gamma \cup \{z_k\}$, $p = 2$, see [17]).

(iv) The theory of the representation of analytic functions by series of polynomials ($F = \Gamma$ or $F = E$, E being a smooth Jordan curve, see [24], [25], [26] .

If we are interested in asymptotics of extremal constants $m_{n,p}(\alpha, F)$ and $T_{n,p}(\alpha, F)$ polynomials, then the cases studied are the following:

1) $F = [-1, +1]$, $d\alpha(x) = \rho(x)dx$, $\rho(x)$ is non negative and integrable. For $p = 2$, we have the classical results of Szegö ([26], [27]). For $1 \leq p \leq \infty$, $\rho(x) = t(x) \big/ \sqrt{1 - x^2}$ and $\log t(x)$ a Riemann integrable function, Berstein [3] found the power asymptotic of the extremal constants $m_{n,p}(\alpha, F)$. Lubinsky and Saff [18] generalized the result of Bernstein by considering $1 \big/ \rho(x) \in L_r[-1, +1]$, $r > 1$.
2) $F = E$, E is a smooth closed Jordan Curve and α is absolutely continuous and satisfies the Szegö condition. The case $0 < p < \infty$ was studied by Gueronimus ([6]). The special case of the unit circle and $p = 2$ has a long history of study (see, for example, [24], [26], [27].).
3) $F = \bigcup_{k=1}^{\varrho} E_k$, E_k being a smooth closed Jordan curve. This case was investigated by Widom ([28]).

For other studies on L_p extremal polynomials see [1], [19], [20], [21].

In this paper we shall study the strong asymptotics of $m_{n,p}(\alpha, F)$ and $T_{n,p}(\alpha, F)$ in the case where $0 < p < \infty$, $F = \Gamma \bigcup \{z_k\}_{k=1}^{\infty}$, Γ being the unit circle, $z_k \in Ext(\Gamma)$, $\alpha = \beta + \gamma$, with $support(\beta) = \Gamma$, $d\alpha = \rho(\xi)\,|d\xi|$ on Γ and $\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k}$ (δ_{z_k} being the Dirac delta unit measure supported at the point z_k).

This work is a generalization of the one of Kaliaguine [9], as well as those of [15] and [12]. In [9] Kaliaguine uses a measure concentrated on a rectifiable Jordan curve plus a finite number of points $\{z_k\}_{k=1}^{l}$. The passage from a finite number to an infinite number of points is a difficult problem and its resolution required, in the case $p = 2$, several years (see [2], [8], [10], [11] and [29]). In ([15], [12]) the authors only consider the relatively simple case $1 \leq p \leq \infty$. We will get in this paper the asymptotic behavior of the L_p extremal polynomials $T_{n,p,\alpha}(z)$, for $0 < p < \infty$.

We give in section 2 some basic definitions and lemmas in the $H^p(G, \rho)$ spaces and define extremal problems on these spaces. Our main result, Theorem 3.1, is stated in section 3.

Let E be a Jordan closed rectifiable curve, $Y = Ext(E)$, $G = \{w \in C : |w| > 1\}$, ($\infty \in Y$, $\infty \in G$) and $w = \Phi(z)$ is the function that maps Y conformally on G in such a manner that $\Phi(\infty) = \infty$ and $\lim_{z \to \infty} (\Phi(z)/z) > 0$. Let Ψ be the inverse function of Φ, $\Psi : G \to Y$.

The method used in this paper is applicable to the case of a contour instead of a circle by considering the functions $\Phi(z)$ and $\Psi(w)$. We shall give a result in our future paper (see [15]).

2. EXTREMAL PROBLEMS IN THE $H^P(G,\rho)$ SPACES

In this section, we introduce some notations and definitions concerning the $H^p(G,\rho)$spaces. Let $G = Ext(\Gamma)$, $G = \{w \in C : |w| > 1\}$, $(\infty \in G)$.

Let $\rho(\xi)$ be an integrable non negative function on Γ. If the weight function $\rho(\xi)$ satisfies the Szegö condition

$$\int_\Gamma (\log \rho(\xi))\,|d\xi| > -\infty, \tag{1}$$

then, one can construct the so-called Szegö function $D_\rho(z)$ associated with the domain G and the weight function $\rho(\xi)$ with the following properties:

$D_\rho(z)$ is analytic in G, $D_\rho(z) \neq 0$ in G, $D_\rho(\infty) > 0$; $D_\rho(z)$ has limit values on Γ and

$$|D_\rho(\xi)|^p = \rho(\xi)\,,\ \xi \in \Gamma\ (a.e.\ on\ \Gamma), \tag{2}$$

where $D_\rho(\xi) = \lim_{z\to\xi} D_\rho(z)\,,\ (a.e.\ on\ \Gamma)$. Define the function D as follows

$$\begin{aligned} D(w) &= D_\rho(\tfrac{1}{w}),\ w \in U \setminus \{0\}\,,\ U = \{w : |w| < 1\}\,, \\ D(0) &= D_\rho(\infty) \end{aligned} \tag{3}$$

One gets a construction and an explicit representation of $D(w)$ in [26]

$$D(w) = \exp\left\{\tfrac{1}{2p\pi}\int_0^{2\pi} \log(\rho(e^{i\theta}))\tfrac{e^{i\theta}+w}{e^{i\theta}-w}d\theta\right\}. \tag{4}$$

Let f be an analytic function in G. For $p > 0$, we say that f belongs to $H^p(G,\rho)$ if $f.D_\rho$ is a function from the space $H^p(G)$. For a function F analytic in G, we say that $F \in H^p(G)$ if and only if $F(1/w) \in H^p(U)$. The space $H^p(U)$ is well known (see [4, 13, 23, 14]).

Each function $f(z)$ from $H^p(G,\rho)$ has limit values on Γ and

$$\|f\|^p_{H^p(G,\rho)} = \int_\Gamma |f(\xi)|^p \rho(\xi)\,|d\xi|\,, \tag{5}$$

where

$$f(\xi) = \lim_{z\to\xi} f(z), \quad \text{a.e. on } \Gamma.$$

If $1 \le p < \infty$, then $H^p(G,\rho)$ is a Banach space with the norm (5). For $0 < p < 1$, $H^p(G,\rho)$ is not a normed space, but it is a complete metric space with the distance

$$d(f,g) = \|f-g\|^p_{H^p(G,\rho)}\,.$$

The following lemmas summarize some properties of the $H^p(G,\rho)$ spaces.

Lemma 2.1. [9] *If $f(z) \in H^p(G,\rho)$ and $K \subset G$, K compact, then there exists a constant $C(K)$ (depending only on K) such that:*

$$\sup_K |f(z)| \le C(K)\,\|f\|^p_{H^p(G,\rho)}\,. \tag{6}$$

Lemma 2.2. [9] *Let* $\{f_n\}$ *be a sequence of functions in* $H^p(G,\rho)$ *and*
(i) $f_n \to f$ *uniformly on the compact sets of* G.
(ii) $\|f_n\|^p_{H^p(G,\rho)} \le M$ *(const.)*.
Then

$$f \in H^p(G,\rho) \ \text{ and } \ \|f\|^p_{H^p(G,\rho)} \le \lim\inf_{n\to\infty} \|f_n\|^p_{H^p(G,\rho)}. \tag{7}$$

For $0 < p < \infty$, define $\mu(\beta)$, $\mu(\alpha)$ and $\mu(\alpha_l)$ as the extremal values of the following extremal problems respectively

$$\mu(\beta) = \inf\left\{\|\varphi\|^p_{H^p(G,\rho)}, \ \ \varphi \in H^p(G,\rho), \ \ \varphi(\infty) = 1\right\}, \tag{8}$$

$$\mu(\alpha) = \inf\left\{\begin{array}{c} \|\varphi\|^p_{H^p(G,\rho)}, \ \varphi \in H^p(G,\rho), \ \varphi(\infty) = 1 \ \text{ and } \varphi(z_k) = 0, \\ k = 1, 2, ... \end{array}\right\}, \tag{9}$$

$$\mu(\alpha_l) = \inf\left\{\begin{array}{c} \|\varphi\|^p_{H^p(G,\rho)}, \ \varphi \in H^p(G,\rho), \ \varphi(\infty) = 1 \text{ and } \varphi(z_k) = 0, \\ k = 1, 2, ..., l \end{array}\right\}, \tag{10}$$

We denote by φ^* and ψ^* the extremal functions of the problems (8) and (9) respectively. We have

Lemma 2.3. ([2], [9], [10]) *The extremal functions* φ^* *and* ψ^* *are connected by*

$$\psi^*(z) = \varphi^*(z). \prod_{k=1}^{\infty} \frac{z - z_k}{z.\overline{z_k} - 1} \cdot \frac{|z_k|^2}{z_k} \tag{11}$$

and

$$\mu(\alpha) = \left(\prod_{k=1}^{\infty} |z_k|\right)^p \cdot \mu(\beta), \tag{12}$$

where

$$\mu(\beta) = \|\varphi^*\|^p_{H^p(G,\rho)} = \left\|\frac{D_\rho(\infty)}{D_\rho}\right\|^p_{H^p(G,\rho)} = [D_\rho(\infty)]^p = [D(0)]^p. \tag{13}$$

3. MAIN RESULTS

We now study the asymptotic behavior of the extremal polynomials $\{T_{n,p,\alpha}(z)\}$. As previously let $\alpha = \beta + \gamma$ be a finite positive measure defined on the Borelian σ-algebra of C and concentrated on the set $F = \Gamma \cup \{z_k\}_{k=1}^{\infty}$, where $\Gamma = \{z : |z| = 1\}$, $\{z_k\}_{k=1}^{\infty}$ be an infinite set of points which lay at the exterior of Γ, β and γ are defined as follows:

β is a measure concentrated on Γ and is absolutely continuous with respect to the Lebesgue measure $|d\xi|$ on the arc, i.e.:

$$d\beta(\xi) = \rho(\xi)\,|d\xi|, \ \ \rho : \Gamma \to \mathbb{R}_+ \ \text{ and } \int_\Gamma \rho(\xi)\,|\mathrm{d}\xi| < +\infty \tag{14}$$

and γ is a discrete measure with masses A_k at the points $z_k \in Ext(\Gamma)$, $k = 1, 2, ...$, i.e.:

$$\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k}, \quad A_k > 0 \ \ and \ \sum_{k=1}^{\infty} A_k < \infty, \tag{15}$$

where δ_{z_k}denotes the (Dirac delta) unit measure supported at the point z_k. By $\mathfrak{P}_{n,1}$ we denote the set of monic polynomials of degree n.

For $0 < p < \infty$, define $m_{n,p}(\alpha)$, $m_{n,p}(\alpha_\ell)$, $m_{n,p}(\beta)$, $T_{n,p,\alpha}(z) = z^n + ... \in \mathfrak{P}_{n,1}$, $T_{n,p,\alpha_\ell}(z) \in \mathfrak{P}_{n,1}$ and $T_{n,p,\beta}(z) \in \mathfrak{P}_{n,1}$ as follows:

$$m_{n,p}(\alpha) = \|T_{n,p,\alpha}\|_{L_p(\alpha,F)} = \inf_{Q_n \in \mathfrak{P}_{n,1}} \|Q_n\|_{L_p(\alpha,F)}, \tag{16}$$

$$m_{n,p}(\alpha_\ell) = \|T_{n,p,\alpha_\ell}\|_{L_p(\alpha_\ell,F_\ell)} = \inf_{Q_n \in \mathfrak{P}_{n,1}} \|Q_n\|_{L_p(\alpha_\ell,F_\ell)},$$

$$m_{n,p}(\beta) = \|T_{n,p,\beta}\|_{L_p(\beta,\Gamma)} = \inf_{Q_n \in \mathfrak{P}_{n,1}} \|Q_n\|_{L_p(\beta,\Gamma)},$$

where

$$\alpha_\ell = \beta + \sum_{k=1}^{\ell} A_k \delta_{z_k} \ and \ F_\ell = \Gamma \cup \{z_k\}_{k=1}^{\ell},$$

Definition 3.1. *Let $\alpha = \beta + \gamma$, we say that the measure α belongs to the class $\mathfrak{BA}$ (denoted by $\alpha \in \mathfrak{BA}$) if the absolute and discrete parts of α, satisfy in addition to the natural relations* (1), (14) *and* (15), *the following conditions*

$$\left(\sum_{k=1}^{\infty} |z_k| - 1\right) < \infty, \tag{17}$$

and

$$\limsup_{n\to\infty} m_{n,p}(\alpha) \le (\mu(\alpha))^{1/p}; \quad 0 < p < \infty \tag{18}$$

Condition (17) guarantees the convergence of the Blaschke product $B_\infty(z)$ associated to the points $\{z_k\}_{k=1}^{\infty}$

$$B_\infty(z) = \prod_{k=1}^{\infty} \frac{z - z_k}{z.\overline{z_k} - 1} \cdot \frac{|z_k|^2}{z_k}. \tag{19}$$

Condition (18) was proven by Khaldi and Benzine [11] and Perherstorfer and Yuditskii ([29], pp. 3217–3219) for the case $p = 2$. To arrive at their results, Khaldi and Benzine used properties of orthogonality of the polynomials $T_{n,2,\alpha}$, but the proof by Perherstorfer and Yuditskii ([29]) is essentially based in many points on the extremal properties of the polynomials $T_{n,2,\alpha}$.

We conclude this section by formulating the main result of this paper.

Theorem 3.1. *If $p > 0$ and $\alpha \in \mathfrak{BA}$, then*

(i) $\lim_{n\to\infty} m_{n,p}(\alpha) = (\mu(\alpha))^{1/p}$

(ii) $\lim_{n\to\infty} \left\| \frac{T_{n,p,\alpha}(z)}{z^n} - \psi^*(z) \right\|_{H^p(G,\rho)} = 0$

(iii) $T_{n,p,\alpha}(z) = z^n [\psi^*(z) + \varepsilon_n(z)]$, $\varepsilon_n(z) \to 0$ *uniformly on the compact sets of* G.

Proof. (i) $p > 0$ *and* $\alpha \in \mathfrak{L}$, *then*

$$\limsup_{n\to\infty} m_{n,p}(\alpha) \leq (\mu(\alpha))^{1/p}. \tag{20}$$

It remains us to show

$$(\mu(\alpha))^{1/p} \leq \liminf_{n\to\infty} m_{n,p}(\alpha). \tag{21}$$

We will present two proofs of this inequality.
First proof of (21).
The extremal property of $T_{n,p,\alpha}(z)$ and $T_{n,p,\alpha_\ell}(z)$ imply

$$\begin{aligned} m_{n,p}(\alpha) = \|T_{n,p,\alpha}\|_{L_p(\alpha,F)} &\geq \|T_{n,p,\alpha}\|_{L_p(\alpha_\ell,F_\ell)} \\ &\geq \|T_{n,p,\alpha_\ell}\|_{L_p(\alpha_\ell,F_\ell)} = m_{n,p}(\alpha_\ell), \end{aligned} \tag{22}$$

(22) implies

$$m_{n,p}(\alpha) \geq m_{n,p}(\alpha_\ell), \quad \forall p > 0, \ \forall \ell. \tag{23}$$

Using this result and theorem 2.2 of [9], we obtain

$$\liminf_{n\to\infty} (m_{n,p}(\alpha)) \geq (\mu(\alpha_\ell))^{1/p}, \ \forall p > 0, \ \forall \ell. \tag{24}$$

Now, using the fact that

$$\mu(\alpha_\ell) = \mu(\beta) \cdot \left(\prod_{k=1}^{\ell} |z_k|\right)^p,$$

(see [9], formula (1.9)) we obtain when $\ell \to \infty$

$$\liminf_{n\to\infty} (m_{n,p}(\alpha)) \geq \cdot\mu(\beta)^{1/p} \cdot \left(\prod_{k=1}^{\infty} |z_k|\right) = (\mu(\alpha))^{1/p}. \tag{25}$$

Second proof of (21).
Putting

$$\phi^*_{n,p} = T_{n,p,\alpha}(z)/z^n, \tag{26}$$

and using (20) we get:

$$\left\|\phi^*_{n,p}\right\|_{H^p(G,\rho)} \leq M = \text{const.} \tag{27}$$

Let $M^* = \liminf_{n\to\infty} \left\|\phi^*_{n,p}\right\|^p_{H^p(G,\rho)}$, we have

$$M^* = \lim_{n\to\infty,\, n\in N_1} \left\|\phi^*_{n,p}\right\|^p_{H^p(G,\rho)}. \tag{28}$$

This result and lemma 2.1 imply that $\left\{\phi^*_{n,p},\ n \in N_1\right\}$ is a normal family in G. So that we can find a function $\psi(z)$ that is the uniform limit (on the compact subsets of G) of some subsequence $\left\{\phi^*_{n,p},\ n \in N_2\right\}$ of $\left\{\phi^*_{n,p},\ n \in N_1\right\}$.

From Lemma 2.2, we get $\psi \in H^p(G,\rho)$ and

$$\|\psi\|^p_{H^p(G,\rho)} \le \lim\inf_{n\to\infty} \left\|\phi^*_{n,p}\right\|^p_{H^p(G,\rho)} . \tag{29}$$

On the other hand $\psi(\infty) = 1$ and $\psi(z_k) = 0$, $k = 1, 2, \ldots$.

We have finally from (26) that

$$\mu(\alpha) \le \|\psi\|^p_{H^p(G,\rho)} \le \lim\inf_{n\to\infty} \left\|\phi^*_{n,p}\right\|^p_{H^p(G,\rho)} \le \lim\inf_{n\to\infty} (m_{n,p}(\alpha))^p , \tag{30}$$

(18) and (30) imply

$$(\mu(\alpha))^{1/p} \le \lim\inf_{n\to\infty} m_{n,p}(\alpha) \le \lim\sup_{n\to\infty} m_{n,p}(\alpha) \le (\mu(\alpha))^{1/p} ,$$

and (i) follows. (ii) For the functions

$$\Psi_n = \tfrac{1}{2}\left(\phi^*_{n,p} + \psi^*\right),$$

where

$$\|\psi^*\|^p_{H^p(G,\rho)} = \mu(\alpha),$$

we have

$$\Psi_n(\infty) = 1 \;\; and \;\; \lim_{n\to\infty} \Psi_n(z_k) = 0, \;\; for\ k = 1, 2, \ldots .$$

As in (i), we have

$$\lim\inf_{n\to\infty} \|\Psi_n\|^p_{H^p(G,\rho)} \ge \mu(\alpha) . \tag{31}$$

Proof of ii), $1 \le p \le \infty$. We obtain ii) by using the Clarkson inequality. More precisely $1 \le p \le 2$.

$$\left[\int_\Gamma \left|\tfrac{1}{2}\left(\phi^*_{n,p} + \psi^*\right)\right|^p \rho(\xi)\,|\mathrm{d}\xi|\right]^{1/p-1} + \left[\int_\Gamma \left|\tfrac{1}{2}\left(\phi^*_{n,p} - \psi^*\right)\right|^p \rho(\xi)\,|\mathrm{d}\xi|\right]^{1/p-1} \le$$
$$\le \left[\tfrac{1}{2}\int_\Gamma \left|\phi^*_{n,p}\right|^p \rho(\xi)\,|d\xi| + \tfrac{1}{2}\int_\Gamma |\psi^*|^p \rho(\xi)\,|\mathrm{d}\xi|\right]^{1/p-1} ,$$

$2 \le p < \infty$,

$$\int_\Gamma \left|\tfrac{1}{2}\left(\phi^*_{n,p} + \psi^*\right)\right|^p \rho(\xi)\,|\mathrm{d}\xi| + \int_\Gamma \left|\tfrac{1}{2}\left(\phi^*_{n,p} - \psi^*\right)\right|^p \rho(\xi)\,|\mathrm{d}\xi| \le$$
$$\le \tfrac{1}{2}\int_\Gamma \left|\phi^*_{n,p}\right|^p \rho(\xi)\,|\mathrm{d}\xi| + \tfrac{1}{2}\int_\Gamma |\psi^*|^p \rho(\xi)\,|\mathrm{d}\xi| .$$

$\mathbf{0 < p < 1}$.

We use the extension of Keldysh theorem (see theorem 2 pp. 430–431 of [1]). More precisely if one notes that in our case the singular part of the measure β is equal to zero and if one takes in consideration the transformation $z \to \frac{1}{z}$, we obtain the following version of theorem 2 of [1].

LEMMA 3.1. *([1]) Let $\{z_k\}_{k=1}^{\infty}$ be a set of points in G, $\alpha = \beta + \gamma$ such that $\alpha \in \mathfrak{BA}$ and $\{f_n\} \subset H^p(G,\rho)$, $0 < p < \infty$. If we note:*

$$\widetilde{f}_n = \frac{f_n}{\varphi^*},\ where\ \varphi^*(z) = \frac{D_\rho(\infty)}{D_\rho(z)},$$

and if

(a) $\lim_{n\to\infty} \widetilde{f}_n(\infty) = 1$,
(b) $\lim_{n\to\infty} \widetilde{f}_n(z_k) = 0,\ k = 1, 2, ...,$
(c) $\sum_{k=1}^{\infty} (|z_k| - 1) < +\infty$,
(d) $\lim_{n\to\infty} \|f_n\|_{H^p(G,\rho)} = D_\rho(\infty) \prod_{k=1}^{\infty} |z_k|$,

then we have

$$\lim_{n\to\infty} \left\| f_n - \prod_{k=1}^{\infty} \frac{z - z_k}{z.\overline{z_k} - 1} \cdot \frac{|z_k|^2}{z_k} .\varphi^* \right\|_{H^p(G,\rho)} = \lim_{n\to\infty} \|f_n - (B_\infty.\varphi^*)\|_{H^p(G,\rho)} = 0.$$

We get (ii) by applying Lemma 3.1 to the sequence $\left\{f_n = \phi^*_{n,p}\right\} \subset H^p(G,\rho)$. Effectively we have:

$$\phi^*_{n,p}(\infty) = 1\ and\ \varphi^*(\infty) = 1,$$

then (a) follows. In the other hand (b) is the consequence of the fact that $\varphi^*(z_k) \neq 0$ and

$$\lim_{n\to\infty} \phi^*_{n,p}(z_k) = 0,\ k = 1, 2, ...;$$

(c) is exactly the condition (17). We obtain (d) by considering (12), (13) and the fact that

$$\lim_{n\to\infty} \left\|\phi^*_{n,p}\right\|_{H^p(G,\rho)} = \lim_{n\to\infty} m_{n,p}(\alpha) = (\mu(\alpha))^{1/p}\ \ (see\ Theorem\ 3.1\ ).$$

(iii) we apply lemma 2.1 for the function

$$\varepsilon_n(z) = \frac{T_{n,p,\alpha}(z)}{z^n} - \psi^*(z). \tag{32}$$

Then for all compact $K \subset G$, we have

$$\sup_{z\in K} |\varepsilon_n(z)| \leq C(K) \|\varepsilon_n\|^p_{H^p(G,\rho)} \to_{n\to\infty} 0. \tag{33}$$

This completes the proof of Theorem 3.1. □

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