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ON THE BOUNDENESS OF THE ASSOCIATED SEQUENCE OF MANN ITERATION FOR SEVERAL OPERATOR CLASSES WITH APPLICATIONS*

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Abstract. We prove that the associated sequence of Mann iteration is decreasing and hence bounded provided that the operator satisfy minimal assumptions. In particular we obtain for a nonexpansive operator that the associated sequence of Ishikawa iteration is decreasing for a nonexpansive operator. Applications to the convergence of Mann iteration are given.

MSC 2000. 47H10.

Keywords. Ishikawa-Mann iteration, associated Mann sequence.

1. PRELIMINARIES

Let X be a real Hilbert space, let $B \subset X$ be a nonempty, convex set. Let $T : B \to B$ be a map. Let $x_1 \in B$, be an arbitrary fixed point. We consider the iteration

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n. \tag{1}$$

The sequence $(\alpha_n)_n$ satisfies:

$$(\alpha_n)_n \subset (0,1),\ \sum_{n=1}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0. \tag{2}$$

Iteration (1) is known as Mann iteration, see [2]. Supposing that T has a unique fixed point x^*, we can associated for (1), a nonnegative sequence $(\|x_n - x^*\|)_n$, named *the associated Mann sequence.* Ishikawa iteration is given by, see [1]:

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n T v_n, \\ v_n &= (1-\beta_n)u_n + \beta_n T u_n. \end{aligned} \tag{3}$$

The sequences $(\alpha_n)_n, (\beta_n)_n$ are in $(0,1)$. Supposing again that T has a unique fixed point x^*, we can associated for Ishikawa iteration, a nonnegative sequence $(\|u_n - x^*\|)_n$. We will name this sequence *the associated Ishikawa sequence.*

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When T is a nonexpansive map, the associated Ishikawa and Mann sequences are decreasing.

Proposition 1. *Let X be a Banach space, let $B \subset X$ be a nonempty, convex set. Let $T : B \to B$ be a nonexpansive map (i.e.$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in B$), then the associated Ishikawa sequence is decreasing.*

Proof. Let $x^* = Tx^*$. For all $x, y \in B$ we have

$$\begin{aligned}
&\|u_{n+1} - x^*\| = \\
&= \|(1-\alpha_n)(u_n - x^*) + \alpha_n(Tv_n - Tx^*)\| \\
&\leq (1-\alpha_n)\|u_n - x^*\| + \alpha_n \|Tv_n - Tx^*\| \\
&\leq (1-\alpha_n)\|u_n - x^*\| + \alpha_n \|v_n - x^*\| \\
&= (1-\alpha_n)\|u_n - x^*\| + \alpha_n \|(1-\beta_n)(u_n - x^*) + \beta_n(Tv_n - Tx^*)\| \\
&\leq (1-\alpha_n)\|u_n - x^*\| + \alpha_n ((1-\beta_n)\|u_n - x^*\| + \beta_n \|Tu_n - Tx^*\|) \\
&\leq (1-\alpha_n)\|u_n - x^*\| + \alpha_n ((1-\beta_n)\|u_n - x^*\| + \beta_n \|u_n - x^*\|) \\
&= (1-\alpha_n)\|u_n - x^*\| + \alpha_n \|u_n - x^*\| \\
&= \|u_n - x^*\|.
\end{aligned}$$

□

Setting $\beta_n = 0, \forall n \in \mathbb{N}$,we get a similar result for Mann iteration.

Proposition 2. *Let X be a Banach space, let $B \subset X$ be a nonempty, convex set. Let $T : B \to B$ be a nonexpansive map (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in B$), then the associated Mann sequence $(\|x_n - x^*\|)_n$ is decreasing.*

We need the following definition:

Definition 3. *The map $T : B \to B$ is called strongly pseudocontractive if there exists $k \in (0,1)$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in B. \tag{4}$$

In a real Hilbert space we have equivalently the following definition: T is strongly pseudocontractive map if there exists $k \in (0,1)$ such that

$$\langle Tx - Ty, x - y\rangle \leq k \|x - y\|^2, \forall x, y \in B. \tag{5}$$

If $k = 1$ then T is a pseudocontractive map. Denote by I the identity map.

Remark 1. A map S is (strongly) accretive if and only if $I - S$ is (strongly) pseudocontractive. □

The aim of this note is to prove that this sequence converges decreasing to zero when T is a strongly pseudocontraction. Also similar results hold for an accretive map and a strongly accretive map. This property will be a good tool, in order to prove the convergence of Mann iteration when the operator is with unbounded range.

The following result is proved in [1].

LEMMA 4. [1] *Let X be a real Hilbert space, the following relation is true for all $x, y \in X$, and for all $\lambda \in (0,1)$:*

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 . \tag{6}$$

We need the following inequality.

LEMMA 5. *If X is a real Hilbert space, then the following relation is true*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in X. \tag{7}$$

Proof. For all $x, y \in X$we have

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2\langle y, x\rangle \leq \|x\|^2 + 2\langle y, y\rangle + 2\langle y, x\rangle \\ &\leq \|x\|^2 + 2\langle y, x+y\rangle .\end{aligned}$$ □

Also we need the convergence of a sequence supplied by an inequality.

LEMMA 6. [4] *Let $(a_n)_n$ be a nonnegative sequence which satisfies the following inequality*

$$a_{n+1} \leq (1-\lambda_n)a_n + \sigma_n, \tag{8}$$

where $\lambda_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

2. MAIN RESULTS

THEOREM 7. *Let X be a real Hilbert space and let $B \subset X$ be a nonempty convex set, and $T : B \to B$ a strongly pseudocontractive operator with constant k. Then the associated Mann sequence $(\|x_n - x^*\|)_n$ is decreasing.*

Proof. Strongly pseudocontractivity assures the uniqueness of the fixed point. Thus the sequence $(\|x_n - x^*\|)_n$ is well defined. Using (6), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Tx_n - Tx^*)\|^2 \\ &= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|Tx_n - x^*\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 .\end{aligned}$$

If there exists the number k_0 such that $\|x_{k_0} - x^*\| = 0$, then $\|x_n - x^*\| = 0, \forall n \geq k_0$.

If $\|x_n - x^*\| \neq 0, \forall n \in N$, then we have

$$\frac{\|x_{n+1}-x^*\|^2}{\|x_n-x^*\|^2} = (1-\alpha_n) + \alpha_n\frac{\|Tx_n-x^*\|^2}{\|x_n-x^*\|^2} - \alpha_n(1-\alpha_n)\frac{\|Tx_n-x_n\|^2}{\|x_n-x^*\|^2} \leq 1.$$

The last inequality is equivalent to

$$-1 + \frac{\|Tx_n-x^*\|^2}{\|x_n-x^*\|^2} - (1-\alpha_n)\frac{\|Tx_n-x_n\|^2}{\|x_n-x^*\|^2} \leq 0 \Leftrightarrow$$

$$\frac{\|Tx_n-x^*\|}{\|x_n-x^*\|^2} \leq 1 + (1-\alpha_n)\frac{\|Tx_n-x_n\|^2}{\|x_n-x^*\|^2}. \tag{9}$$

The proof is complete if we are able to prove (9). From strongly pseudocontractivity we have

$$\frac{\|Tx_n-x^*\|^2}{\|x_n-x^*\|^2} \le 1 + k\frac{\|Tx_n-x_n\|^2}{\|x_n-x^*\|^2}.$$

Because $\lim_{n\to\infty} \alpha_n = 0$, there exists n_0 such that for all $n \ge n_0$ we have $\alpha_n \le 1-k$. Thus, we get

$$1 + k\frac{\|Tx_n-x_n\|^2}{\|x_n-x^*\|^2} \le 1 + (1-\alpha_n)\frac{\|Tx_n-x_n\|^2}{\|x_n-x^*\|^2}.$$

Hence (9) is true. □

Consider the following equation

$$Sx = f,$$

where S is a strongly accretive map and f is a given point in X. Consider the map $Tx = f + (I-S)x,\ \forall x \in X$. A fixed point for T will be a solution for the equation $Sx = f$. Remark 1 and Theorem 7 lead to the following result.

Corollary 8. *Let X be a real Hilbert space and $S : X \to X$ a strongly accretive operator with constant k. Then the associated Mann sequence $(\|x_n - x^*\|)_n$ is decreasing, where $(x_n)_n$ is given by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\left(f + (I-S)\right)x_n,$$

with $(\alpha_n)_n$ satisfying (2), *where x^* is a solution for $Sx = f$.*

Consider the following equation

$$x + Sx = f,$$

where S is an accretive map and f is a given point. Consider the map $Tx = f - Sx,\ \forall x \in X$. A fixed point for T will be a solution for the equation $Sx = f$.

Remark 2. [3] If S is accretive map then $T = f - S$ is strongly pseudocontractive map. □

Remark 2 and Theorem 7 lead to the following result.

Corollary 9. *Let X be a real Hilbert space and $S : X \to X$ an accretive operator. Then the associated Mann sequence $(\|x_n - x^*\|)_n$ is decreasing, where $(x_n)_n$ is given by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\left(f - S\right)x_n,$$

with $(\alpha_n)_n$ satisfying (2).

3. APPLICATIONS TO THE CONVERGENCE OF MANN ITERATION

THEOREM 10. *Let X be a real Hilbert space and $B \subset X$ a nonempty convex closed set. Let $T : B \to B$ be a completely continuous and strongly pseudocontractive map with fixed points. Then $(x)_n$ given by (1) strongly converges to the fixed point of T namely x^*.*

Proof. Strongly pseudocontractivity assures the uniqueness of the fixed point. Inequalities (7) and (5) lead us to,

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 = \\
&= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Tx_n - x^*)\|^2 \\
&\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Tx_n - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle Tx_n - x^*, x_n - x^*\rangle \\
&\quad + 2\alpha_n \langle Tx_n - x^*, (x_{n+1} - x^*) - (x_n - x^*)\rangle \\
&\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n k \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \langle Tx_n - x^*, x_{n+1} - x_n\rangle \\
&\le \left(1 - 2(1-k)\alpha_n + \alpha_n^2\right) \|x_n - x^*\|^2 + 2\alpha_n \|Tx_n - x^*\| \|x_{n+1} - x_n\| \\
&\le \left(1 - 2(1-k)\alpha_n + \alpha_n^2\right) \|x_n - x^*\|^2 + 2\alpha_n (\|Tx_n\| + \|x^*\|)\, \alpha_n \|Tx_n - x_n\| \\
&\le \left(1 - 2(1-k)\alpha_n + \alpha_n^2\right) \|x_n - x^*\|^2 + 2\alpha_n^2 (\|Tx_n\| + \|x^*\|)(\|Tx_n\| + \|x_n\|) .
\end{aligned}$$

The map T is strongly pseudocontractive. Theorem 7 assures that $(\|x_n - x^*\|)_n$ is decreasing, hence it is bounded, i.e. $(x_n)_n$ is bounded. The map T is completely continuous and $(x_n)_n$ is bounded, so $(\|Tx_n\|)_n$ is bounded. Thus, there exists a strict positive $M > 0$ such that

$$\|x_{n+1} - x^*\|^2 \le \left(1 - 2(1-k)\alpha_n + \alpha_n^2\right) \|x_n - x^*\|^2 + \alpha_n^2 M.$$

Condition $\lim_{n\to\infty} \alpha_n = 0$ assures the existence of a rank n_0 such that for all $n \ge n_0$, we have $(1-k) \ge \alpha_n$, thus

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \left(1 - 2(1-k)\alpha_n + \alpha_n^2\right) \|x_n - x^*\|^2 + \alpha_n^2 M \\
&\le (1 - 2(1-k)\alpha_n + (1-k)\alpha_n) \|x_n - x^*\|^2 + \alpha_n^2 M \\
&= (1 - (1-k)\alpha_n) \|x_n - x^*\|^2 + \alpha_n^2 M.
\end{aligned}$$

Denote by

$$\begin{aligned}
a_n &:= \|x_n - x^*\|^2 , \\
\lambda_n &:= (1-k)\alpha_n \in (0,1), \\
\sigma_n &:= \alpha_n^2 M.
\end{aligned}$$

Remark that $\sigma_n = o(\lambda_n)$ and inequality (8) is satisfied. From Lemma 6, we get $\lim_{n\to\infty} a_n = 0$, that is $\lim_{n\to\infty} x_n = x^*$. □

Corollary 8 and Theorem 10 lead to the following result.

Corollary 11. *Let X be a real Hilbert space and $B \subset X$ a nonempty convex closed set. Let $T : B \to B$ be a completely continuous and strongly accretive map with fixed points. Then $(x)_n$ given by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \left(f + (I - S)\right) x_n,$$

with $(\alpha_n)_n$ satisfying (2), strongly converges to the solution of $Sx = f$.

Corollary 9 and Theorem 10 lead to the following result.

Corollary 12. *Let X be a real Hilbert space and $B \subset X$ a nonempty convex closed set. Let $T : B \to B$ be a completely continuous and accretive map with fixed points. Then $(x)_n$ given by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \left(f - S\right) x_n,$$

with $(\alpha_n)_n$ satisfying (2), strongly converges to the solution of $x + Sx = f$.

Remark 3. Usually, for the convergence of Mann iteration, either the set B or the sequence $(\|Tx_n\|)_n$ is bounded. From the above results one can see that in a Hilbert space both assumptions are redundant. □

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