REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
**Rev. Anal. Numér. Théor. Approx., vol. 34 (2005) no. 2, pp. 135–138**
ictp.acad.ro/jnaat

# ON THE MODIFIED BETA APPROXIMATING OPERATORS OF SECOND KIND

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**Abstract.** We shall define a general linear operator from which we obtain as particular case the modified beta second kind operator

$$(T_{p,q}f)(x) = \tfrac{1}{B(p,q)} \int_0^\infty \tfrac{u^{p-1}}{(1+u)^{p+q}} f\left(\tfrac{B(p,q)}{B(p+a,q-a)} u^a x\right) \mathrm{d}u.$$

We consider here only the case $a = 1$.

We obtain several positive linear operators as a particular case of this modified beta second kind operator.

**MSC 2000.** 41A36.
**Keywords.** Euler's beta function, the modified beta second kind operator, positive linear operators.

## 1. INTRODUCTION

Euler's beta function of second kind is defined for $p > 0$, $q > 0$, by the following formula

$$B(p,q) = \int_0^\infty \tfrac{u^{p-1}}{(1+u)^{p+q}} \mathrm{d}u. \tag{1}$$

The beta transform of the function $f$ is defined by the following formula [6]

$$\mathcal{B}_{p,q} f = \tfrac{1}{B(p,q)} \int_0^\infty \tfrac{u^{p-1}}{(1+u)^{p+q}} f(u) \mathrm{d}u. \tag{2}$$

The modified beta operator is defined for $x \geq 0$ by the following formula

$$(\mathcal{B}_{p,q} f)(x) = \tfrac{1}{B(p,q)} \int_0^\infty \tfrac{u^{p-1}}{(1+u)^{p+q}} f(ux) \mathrm{d}u. \tag{3}$$

We shall define a more general linear operator from which we obtain as a particular case the modified beta second kind operator.

For $a, b \in \mathbb{R}$ and $x \geq 0$ we define the $(a,b)$-modified beta operator

$$(\mathcal{B}_{p,q}^{(a,b)} f)(x) = \tfrac{1}{B(p,q)} \int_0^\infty \tfrac{u^{p-1}}{(1+u)^{p+q}} f\left(\tfrac{B(p,q)}{B(p+a,q+b)} \cdot \tfrac{u^a x}{(1+u)^{a+b}}\right) \mathrm{d}u, \tag{4}$$

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where $B(\cdot,\cdot)$ is the beta function (1) and $f$ is any real measurable function defined on $(0,\infty)$ such that

$$(\mathcal{B}_{p,q}^{(a,b)}|f|)(x) < \infty.$$

## 2. THE MODIFIED BETA SECOND KIND OPERATOR. CASE $a = 1$

Let us denote by $M[0,\infty)$ the linear space of functions defined for $t \geq 0$, bounded and Lebesgue measurable in each interval $[c,d]$, where $0 < c < d < \infty$.

For $a + b = 0$ we obtain from (4) the modified beta second kind operator

$$(T_{p,q}^{(a)}f)(x) = (\mathcal{B}_{p,q}^{(a,-a)}f)(x) = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{B(p,q)}{B(p+a,q-a)}u^a x\right) \mathrm{d}u \tag{5}$$

for $f \in M[0,\infty)$ such that $(T_{p,q}^{(a)}|f|)(x) < \infty$.

One observes that $T_{p,q}^{(a)}$ is a positive linear operator and $(T_{p,q}^{(a)}(e_1)(x) = x$.

We consider here only the case $a = 1$.

If we choose in (5) $a = 1$ we obtain the modified beta second kind operator

$$(T_{p,q}f)(x) = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{q-1}{p}ux\right) \mathrm{d}u. \tag{6}$$

Remark 1. For $p = q = n$ we obtain the operator

$$(B_n f)(x) = \frac{1}{B(n,n)} \int_0^\infty \frac{u^{n-1}}{(1+u)^{2n}} f\left(\frac{n-1}{n}tx\right) \mathrm{d}t \tag{7}$$

considered by V. Totik in [7]. □

Remark 2. For $\frac{p}{q-1} = x$ or $p = (q-1)x$, $x > 0$, we obtain the operator

$$(T_q f)(x) = \frac{1}{B((q-1)x,q)} \int_0^\infty \frac{u^{(q-1)x-1}}{(1+u)^{(q-1)(x+1)}} f(u)\mathrm{d}u, \quad q > 2$$

considered by the author in [4]. □

Remark 3. If we put $a = -1$ in (5) we obtain the following modified beta second kind operator

$$(T_{p,q}^{(-1)}f)(x) = (\mathcal{T}_{p,q}f)(x) = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{p-1}{q}\frac{x}{u}\right) \mathrm{d}u. \tag{8}$$

Denote $v = 1/u$, that is $u = 1/v$ and $du = -dv/v^2$. We obtain

$$\begin{aligned}(\mathcal{T}_{p,q}f)(x) &= \frac{1}{B(p,q)} \int_0^\infty \frac{(1/v)^{p-1}}{(1+1/v)^{p+q}} f\left(\frac{p-1}{q}xv\right) \frac{\mathrm{d}v}{v^2} \\ &= \frac{1}{B(p,q)} \int_0^\infty \frac{v^{p+q}}{v^{p+1}(1+v)^{p+q}} f\left(\frac{p-1}{q}vx\right) \mathrm{d}v \\ &= \frac{1}{B(p,q)} \int_0^\infty \frac{v^{q-1}}{(1+v)^{p+q}} f\left(\frac{p-1}{q}vx\right) \mathrm{d}v\end{aligned}$$

and we observe that $\mathcal{T}_{p,q}$ is similar with $T_{p,q}$. □

EXAMPLE 1. A double sequence of positive linear operators (names beta operators by Uprety [8], see also [1]) defined as

$$(F_{m,n}f)(x) = \frac{1}{B(m,n)} \int_0^{\infty} \frac{x^n u^{n-1}}{(1+ux)^{m+n}} f\left(\frac{n}{mu}\right) \mathrm{d}u$$

or, equivalently,

$$(F_{m,n}f)(x) = \frac{1}{B(m,n)} \int_0^{\infty} \frac{u^{n-1}}{(1+u)^{m+n}} f\left(\frac{nx}{mu}\right) \mathrm{d}u, \tag{9}$$

$$(F_{m,n}f)(x) = \frac{1}{B(m,n)} \int_0^{\infty} \frac{u^{m-1}}{(1+u)^{m+n}} f\left(\frac{n}{m}xu\right) \mathrm{d}u \tag{10}$$

is similar with (8), and respectively with (6).

THEOREM 1. *The moment of order $k$ $(1 \le k < q)$ of the operator $T_{p,q}$ has the following value*

$$\begin{aligned}
(T_{p,q}e_k)(x) &= \left(\tfrac{q-1}{p}\right)^k \tfrac{p(p+1)\dots(p+k-1)}{(q-1)\dots(q-k)}, \quad 1 \le k < q,\\
(T_{p,q}e_1)(x) &= x; \quad (T_{p,q}e_2)(x) = \tfrac{(p+1)(q-1)}{p(q-2)} x^2, \quad q > 2,\\
T_{p,q}((t-x)^2; x) &= \tfrac{p+q-1}{p(q-2)} x^2, \quad q > 2.
\end{aligned}$$

*Proof.*

$$\begin{aligned}
(T_{p,q}e_k)(x) &= \frac{1}{B(p,q)} \int_0^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}} \left(\frac{q-1}{p}\right)^k u^k x^k \mathrm{d}u\\
&= \left(\frac{q-1}{p}\right)^k \frac{x^k}{B(p,q)} \int_0^{\infty} \frac{u^{p+k-1}}{(1+u)^{p+q}} \mathrm{d}u\\
&= \left(\frac{q-1}{p}\right)^k \frac{B(p+k,q-k)}{B(p,q)} x^k\\
&= \left(\frac{q-1}{p}\right)^k \frac{\Gamma(p+k)\Gamma(q-k)}{\Gamma(p+q)} \cdot \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^k\\
&= \left(\frac{q-1}{p}\right)^k \frac{p(p+1)\dots(p+k-1)}{(q-1)\dots(q-k)} x^k.
\end{aligned}$$

For $k = 1, 2$, we obtain $(T_{p,q}e_1)(x) = x$,

$$(T_{p,q}e_2)(x) = \left(\frac{q-1}{p}\right)^2 \frac{2p(p+1)}{(q-1)(q-2)} x^2 = \frac{(p+1)(q-1)}{p(q-2)} x^2$$

and

$$T_{p,q}((t-x)^2; x) = \left(\frac{(p+1)(q-1)}{p(q-2)} - 1\right) x^2 = \frac{pq-p+q-1-pq+2p}{p(q-2)} x^2 = \frac{p+q-1}{p(q-2)} x^2. \quad \square$$

**Particular case**

For $q = p + 1$ we obtain the positive linear operator

$$(T_p f)(x) = \frac{1}{B(p,p+1)} \int_0^{\infty} \frac{n^{p-1}}{(1+u)^{2p+1}} f(ux) \mathrm{d}u \tag{11}$$

considered by S. M. Mazhar [2] for $p = n$ (see also [3]).

Corollary 2. *The moment of order $k$ $(1 \leq k < p+1)$ of the operator $T_p$ has the following value*

$$(T_p e_k)(x) = \tfrac{(p+1)\dots(p+k-1)}{(p-1)\dots(p-k+1)} x^k, \quad 1 \leq k < p+1,$$
$$(T_p e_2)(x) = \tfrac{p+1}{p-1} x^2, \quad T_p((t-x)^2; x) = \tfrac{2x^2}{p-1}.$$

*Proof.* It is obtained from Theorem 1 for $q = p+1$. □

Remark 4. For $p = n$ we obtain

$$T_n((t-x)^2; x) = \tfrac{2x^2}{n-1}. \qquad \square$$

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Received by the editors: February 18, 2004.