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ON THE EXTENSION OF SEMI-LIPSCHITZ FUNCTIONS ON ASYMMETRIC NORMED SPACES

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Abstract. Extension theorems for semi-Lipschitz functions and some properties of these extensions useful in approximation problems are presented. As illustration, a such problem is considered.

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1. PRELIMINARIES

Let X be a real linear space. A function $p : X \to [0, \infty)$ is called an *asymmetric seminorm* [2] if the following conditions hold for all $x, y \in X$:

(AN1) $$p(x) \geq 0,$$

(AN2) $$p(tx) = tp(x), \quad t \geq 0,$$

(AN3) $$p(x+y) \leq p(x) + p(y).$$

The function $\overline{p} : X \to [0, \infty)$ defined by $\overline{p}(x) = p(-x), x \in X$, is also an asymmetric seminorm on X, called the *conjugate* seminorm to p. The functional

$$p^s(x) = \max\{p(x), p(-x)\}, \quad x \in X,$$

is a seminorm on X. If p^s is a norm on X, then p is called an *asymmetric norm.* The term "asymmetric" is motivated by the fact that it is possible that $p(x) \neq p(-x)$ for some $x \in X$.

A pair (X, p) where X is a real linear space and p an asymmetric seminorm on X, is called an asymmetric seminormed space, respectively asymmetric normed space if p is an asymmetric norm on X. Some properties of such spaces are given in [3], [4], and the references therein.

An example is the following: let $\mathbb{R}$ be the set of the real numbers and $u : \mathbb{R} \to [0, \infty)$, $u(a) = \max\{a, 0\}, a \in \mathbb{R}$. Then the function u is an asymmetric norm on $\mathbb{R}$. The conjugate $\overline{u} : \mathbb{R} \to [0, \infty)$, $\overline{u}(a) = u(-a), a \in \mathbb{R}$, is another

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asymmetric norm on $\mathbb{R}$, because the function $u^s(a) = \max\{u(a), u(-a)\} = |a|$, $a \in \mathbb{R}$, is a norm on $\mathbb{R}$ (see [4], [5]).

If (X, p) is an asymmetric seminormed space, one considers the following topologies on X:

(1) The topology τ_p generated by the families of *forward* open balls $B^+(x, \varepsilon) = \{z \in X : p(z - x) < \varepsilon\}, x \in X, \varepsilon > 0$;
(2) The topology $\tau_{\overline{p}}$ generated by the families of *backward* open balls $B^-(x, \varepsilon) = \{z \in X : p(x - z) < \varepsilon\}, x \in X, \varepsilon > 0$;
(3) The topology τ generated by the open balls $B(x, \varepsilon) = \{z \in X : p^s(x - z) < \varepsilon\} = B^+(x, \varepsilon) \cap B^-(x, \varepsilon), x \in X, \varepsilon > 0$.

For details on these topologies, see [1] and [3].

2. THE CONES OF SEMI-LIPSCHITZ FUNCTIONS

Let (X, p) be an asymmetric seminormed space and Y a subset of X. A function $f : Y \to \mathbb{R}$ is called *p-semi-Lipschitz* if there exists a constant $K \geq 0$ such that

$$f(u) - f(v) \leq K\, p(u - v), \tag{1}$$

for all $u, v \in X$, (see [13]). Observe that a p-semi-Lipschitz function is upper semicontinuous.

The set of all p-semi-Lipschitz functions on Y is denoted by symbol p-SLip$Y$, i.e.

$$p\text{-SLip}Y := \{f : Y \to \mathbb{R},\ f \text{ is p-semi-Lipschitz}\}. \tag{2}$$

If $f \in p$-SLip$Y$, then the nonnegative number

$$||f|_p = \sup\left\{\frac{(f(u) - f(v)) \vee 0}{p(u - v)} : u, v \in Y, p(u - v) > 0\right\} \tag{3}$$

is the smallest p-semi-Lipschitz constant for F, i.e. the following inequality holds:

$$f(u) - f(v) \leq ||f|_p\, p(u - v),$$

for all $u, v \in Y$ (see [10]).

The set p-SLip$Y$ is closed with respect to pointwise addition of functions, and with respect to multiplication of a function by nonnegative scalar, i.e. p-SLip$Y$ is a *cone*.

The functional $||\cdot|_p : p$-SLip$Y \to [0, \infty)$ defined by (3) verifies the properties $(AN1)$–$(AN3)$ of an asymmetric seminorm.

Analogously, a function $f : Y \to \mathbb{R}$ is called $\bar{p}$-semi-Lipschitz if there exists $Q \geq 0$ such that

$$f(u) - f(v) \leq Q\, \bar{p}(u - v) = Q\, p(v - u), \tag{4}$$

for all $u, v \in Y$.

The smallest constant Q in (4) is denoted by $||f|_{\overline{p}}$ and the set of all $\bar{p}$-semi-Lipschitz functions on Y is denoted by symbol $\bar{p}$-SLip$Y$. The functional

$|| \cdot |_{\overline{p}}$:$\bar{p}$-SLip$Y\to$ $[0,\infty)$ defined by an expression analogue with (3) is an asymmetric norm on the cone $\bar{p}$-SLip$Y$.

The function $f : Y \to \mathbb{R}$ is called p^s-*Lipschitz* if there exits $M \geq 0$ such that

$$|f(u) - f(v)| \leq M\, p^s(u-v)\,, \tag{5}$$

for all $u, v \in Y$.

The set of all p^s-Lipschitz functions on Y is denoted by symbol p^s-SLip$Y$. With respect to pointwise addition of functions and multiplication of functions by real numbers, the set p^s-SLip$Y$ is a linear space.

The functional $\|\cdot\|$:p^s-SLip$Y\to$ $[0,\infty)$,

$$\|f\| = \sup\left\{\frac{|f(u)-f(v)|}{p^s(u-v)} : u, v \in Y, p^s(u-v) > 0\right\} \tag{6}$$

is a seminorm on linear space p^s-SLip$Y$.

Proposition 1. *Let (X, p) be an asymmetric seminormed space and Y a subset of X. Then*

a) *The sets p-SLip$Y$ and $\bar{p}$-SLip$Y$ are convex cones in the space p^s-SLip$Y$;*
b) *The function f is in p-SLip$Y$ if and only if $-f$ is in $\bar{p}$-SLip$Y$. For every $f \in p$-SLip$Y$, the following equality holds:*

$$||f|_p = || - f|_{\overline{p}} = \|f\|.$$

Proof. a) Let f in p-SLip$Y$. Then

$$f(u) - f(v) \leq ||f|_p\, p(u-v) \leq ||f|_p\, p^s(u-v)\,,$$

for all $u, v \in Y$.

Changing the place of u and v, we obtain

$$f(v) - f(u) \leq ||f|_p\, p(v-u) = ||f|_p\, \overline{p}(u-v) \leq ||f|_p\, p^s(u-v)\,,$$

for all $u, v \in Y$. It follows that

$$|f(u) - f(v)| \leq ||f|_p\, p^s(u-v)\,,$$

for all $u, v \in Y$. Consequently $f \in p^s$-SLip$Y$ (and $\|f\| = ||f|_p$).

If $f \in \bar{p}$-SLip$Y$, one obtains

$$|f(u) - f(v)| \leq ||f|_p\, p^s(u-v)\,,$$

for all $u, v \in Y$, i.e. $f \in \bar{p}$-SLip$Y$, (it follows $\|f\| = ||f|_{\overline{p}}$).

b) Let $f \in p$-SLip$Y$. By $f(u) - f(v) \leq ||f|_p\; p(u-v)$, for all $u, v \in Y$, it follows

$$(-f)(v) - (-f)(u) \leq ||f|_p\, p(u-v) = ||f|_p\, \overline{p}(v-u)\,,$$

and then $-f \in \bar{p}$-SLip$Y$. Moreover, it follows that $||-f|_{\overline{p}} \leq ||f|_p$. Analogously, $f \in \bar{p}$-SLip$Y$ implies

$$(-f)(v) - (-f)(u) \leq ||f|_{\overline{p}}\, \overline{p}(u-v) = ||f|_{\overline{p}}\, p(v-u)\,,$$

for all $u, v \in Y$. Consequently $-f \in p$-SLip$Y$ and $|| - f|_p \leq ||f|_{\overline{p}}$.

Observe that for $f \in p$-SLip$Y$,

$$||f|_p = || - (-f)\,|_p \leq || - f|_{\overline{p}},$$

and for $f \in \bar{p}$-SLip$Y$, one obtains

$$||f|_{\overline{p}} = || - (-f)\,|_{\overline{p}} \leq || - f|_p.$$

Finally, it follows

$$||f|_p = || - f|_{\overline{p}},\ \forall f \in p\text{-SLip}Y,$$

and

$$||f|_{\overline{p}} = || - f|_p,\ \forall f \in \bar{p}\text{-SLip}Y. \qquad \square$$

Remarks. 1) By Proposition 1, a) it follows that for $f \in p$-SLip$Y$, ($g \in \bar{p}$-SLip$Y$) one obtains $||f|_p = \|f\|$, (respectively $||g|_{\overline{p}} = \|g\|$) and consequently

$$f \in p\text{-SLip}Y \cap \bar{p}\text{-SLip}Y \Rightarrow ||f|_p = ||f|_{\overline{p}} = \|f\|\,.$$

2) Let y_0 be a fixed element in Y. Denote

$$p^s\text{-SLip}_0Y := \{f \in p^s\text{-Lip}Y,\ f\,(y_0) = 0\}\,,$$
$$p\text{-SLip}_0Y := \{f \in p\text{-SLip}Y,\ f\,(y_0) = 0\}\,,$$
$$\bar{p}\text{-SLip}_0Y := \{f \in \bar{p}\text{-SLip}Y,\ f\,(y_0) = 0\}\,.$$

If $f \in p^s$-SLip$_0Y$, then $\|f\| = 0$ implies $f \equiv 0$. It follows that $f \in p$-SLip$_0Y \cap$ $\bar{p}$-SLip$_0Y$ implies $f \equiv 0$ and $||f|_p$, $||f|_{\overline{p}}$ are asymmetric norms on p-SLip$_0Y$ and $\bar{p}$-SLip$_0Y$ respectively.

3) Let Y be a subspace of asymmetric normed space (X, p) and $\varphi : Y \to \mathbb{R}$ a linear functional. If $(\mathbb{R}, u)$ is the asymmetric normed space with asymmetric norm $u\,(a) = \max\{a, 0\},\ a \in \mathbb{R}$ and τ_u is the topology associated to u, then the functional $\varphi : Y \to \mathbb{R}$ is called (p, u)-continuous if it is continuous in the topologies τ_p and τ_u. $\square$

The linear functional $\varphi : (Y, p) \to (\mathbb{R}, u)$ is called *p-bounded* if there exists $L \geq 0$ such that

$$\varphi\,(y) \leq L\,p\,(y)\,,$$

for every $y \in Y$.

The functional $\varphi : (Y, p) \to (\mathbb{R}, u)$ is (p, u)-continuous if and only if φ is p-bounded, (see [4]).

Every p-bounded linear functional on Y is p-semi-Lipschitz on Y.

Denote by Y_p^b the set of all p-bounded functionals on Y. Then $Y_p^b \subset p$-SLip$_0Y$ (here $y_0 = 0$), and for $\varphi \in Y_p^b$,

$$||\varphi|_p = \sup_{y \neq 0} \frac{\varphi\,(y)}{p\,(y)} = \sup\{\varphi\,(y) : y \in Y, p\,(y) \leq 1\}\,. \tag{7}$$

The functional $|| \cdot |_p : Y_p^b \to [0, \infty)$ defined by (7) is an asymmetric norm, and the pair $\left(Y_p^b, || \cdot |_p\right)$ is called *asymmetric dual cone of* (Y, p) [5].

3. EXTENSION RESULTS

Let Y be a subset of asymmetric normed space (X,p). The function $f : Y \to \mathbb{R}$ is called *bounded* on Y if there exists the numbers $m, M \in \mathbb{R}$ such that

$$m \leq f(y) \leq M,$$

for every $y \in Y$.

Proposition 2. *Let (X,p) be an asymmetric normed space, Y a subset of X and $f : Y \to \mathbb{R}$ a bounded function. Let also $K \geq 0$ be arbitrary, but fixed. Then the functions $F_p(f), G_p(f) : X \to \mathbb{R}$ defined by the formulas*

$$F_p(f)(x) = \inf_{y\in Y} \{f(y) + K\, p(x-y)\}, \quad x \in X,$$

$$G_p(f)(x) = \sup_{y\in Y} \{f(y) - K\, p(y-x)\}, \quad x \in X,$$

satisfy the following relations:

(a) $$F_p(f)(y) \leq f(y) \leq G_p(f)(y), \quad \forall y \in Y,$$

(b) $$F_p(f)(x_1) - F_p(f)(x_2) \leq K\, p(x_1 - x_2), \quad \forall x_1, x_2 \in X,$$

(c) $$G_p(f)(x_1) - G_p(f)(x_2) \leq K\, p(x_1 - x_2), \quad \forall x_1, x_2 \in X.$$

Proof. Let $f : Y \to \mathbb{R}$ and $m,\ M \in \mathbb{R}$ such that $m \leq f(y) \leq M$ for every $y \in Y$. It follows

$$m \leq f(y) + K\, p(x-y), \forall y \in Y,\ \forall x \in X,$$

and

$$M \geq f(y) - K\, p(y-x), \ \forall y \in Y,\ \forall x \in X.$$

Consequently, the set $\{f(y) + K\, p(x-y) \mid y \in Y\}$ is bounded from below for every $x \in X$ and the set $\{f(y) - K\, p(y-x) \,|y \in Y\}$ is bounded from above for every $x \in X$. Then the functions $F_p(f)$ and $G_p(f)$ are well defined on X.

a) Let $y_0 \in Y$ be fixed. For every $x \in X$,

$$\inf_{y\in Y} \{f(y) + K\, p(x-y)\} \leq f(y_0) + Kp(x-y_0),$$

and for $x = y_0$ it follows that

$$F_p(f)(y_0) \leq f(y_0).$$

Analogously,

$$\sup_{y\in Y} \{f(y) - K\, p(y-x)\} \geq f(y_0) - K\, p(y_0 - x),$$

and for $x = y_0$, we obtain

$$f(y_0) \leq G_p(f)(y_0).$$

Because y_0 is arbitrary in Y it follows that

$$F_p(f)(y) \leq f(y) \leq G_p(f)(y),$$

for every $y \in Y$.

b) Let $x_1, x_2 \in X$. Because for every $y \in Y$ we have

$$\begin{aligned} 0 &\le p(x_1 - y) \\ &= p(x_1 - x_2 + x_2 - y) \\ &\le p(x_1 - x_2) + p(x_2 - y), \end{aligned}$$

it follows the inequality

$$f(y) + K\, p(x_1 - y) \le f(y) + K\, p(x_1 - x_2) + K\, p(x_2 - y),$$

and taking the infimum with respect to $y \in Y$ one obtain

$$F_p(f)(x_1) \le F_p(f)(x_2) + K\, p(x_1 - x_2).$$

Then

$$F_p(f)(x_1) - F_p(f)(x_2) \le K\, p(x_1 - x_2),$$

for every $x_1, x_2 \in X$.

c) Using the inequalities

$$\begin{aligned} 0 \le p(y - x_2) = p(y - x_1 + x_1 - x_2) \le \\ \le p(y - x_1) + p(x_1 - x_2), \end{aligned}$$

one obtains

$$f(y) - K\, p(y - x_2) \ge f(y) - K\, p(y - x_1) - K\, p(x_1 - x_2)$$

and taking the supremum with respect to $y \in Y$ we obtain

$$G_p(f)(x_1) - G_p(f)(x_2) \le K\, p(x_1 - x_2).$$ □

A subset Y of an asymmetric normed space (X, p) is called *p-bounded*, (*$\overline{p}$-bounded*) if there exists $M \ge 0$ $(Q \ge 0)$ such that

$$p(u - v) \le M \quad (\overline{p}(u - v) \le Q),$$

for all $u, v \in Y$.

The set Y is called $(p, \overline{p})$-*bounded* if it is both p-bounded and $\overline{p}$-bounded.

Proposition 3. *If Y is a $(p, \overline{p})$-bounded set of an asymmetric normed space (X, p) and $f \in p$-SLip$Y$, then f is bounded.*

Proof. Let $y_0 \in Y$ be fixed. For every $y \in Y$, $f(y) - f(y_0) \le ||f|_p\, p(y - y_0)$, implies $f(y) \le f(y_0) + ||f|_p\, p(y - y_0)$.

Analogously,

$$f(y_0) - f(y) \le ||f|_p\, p(y_0 - y) = ||f|_p\, \bar{p}(y - y_0),$$

for every $y \in Y$. Then

$$f(y_0) - ||f|_p\, \overline{p}(y - y_0) \le f(y) \le f(y_0) + ||f|_p\, p(y - y_0),$$

and because Y is $(p, \overline{p})$ −bounded, it follows that f is bounded. □

Remark 1. By Proposition 1, it follows that every $g \in \bar{p}$-SLip$Y$ is bounded, if Y is $(p, \overline{p})$-bounded. For the symmetric case of Proposition 2, see [12]. □

PROPOSITION 4. *Let (X, p) be an asymmetric normed space, Y a $(p, \overline{p})$-bounded subset of X and $f \in p$-SLip$Y$. Let $||f|_p$ the smallest semi-Lipschitz constant of f. Then the functions $F_p(f), G_p(f) : X \to \mathbb{R}$ defined as in Proposition 2 with $K = ||f|_p$ satisfy the following properties:*

a) $G_p(f)(x) \leq F_p(f)(x), x \in X$,

b) $G_p(f)(y) = f(y) = F_p(f)(y), y \in Y$,

c) $||G_p(f)|_p = ||f|_p = ||G_p(f)|_p$,

d) *If $g : X \to \mathbb{R}$ and $g \in p$-SLip$X$ verifies $g|_Y = f$ and $||f|_p = ||g|_p$, then*

$$G_p(f)(x) \leq g(x) \leq F_p(f)(x),$$

for every $x \in X$.

Proof. a) For Every $u, v \in Y$ and $x \in X$, the following inequalities are fulfilled:

$$f(u) - f(v) \leq ||f|_p\, p(u - v) \leq ||f|_p\, p(u - x) + ||f|_p\, p(x - v).$$

Then

$$f(u) - ||f|_p\, p(u - x) \leq f(v) + p(x - v).$$

Taking the infimum with respect to $v \in Y$ and then with respect to $u \in Y$, we obtain

$$G_p(f)(x) \leq F_p(f)(x),$$

for every $x \in X$.

b) It follows from Proposition 2 a), Proposition 3 and the previous inequality of a).

c) The following inequalities are obvious:

$$||G_p(f)|_p \geq ||f|_p \text{ and } ||F_p(f)|_p \geq ||f|_p.$$

Let now $x_1, x_2 \in X$ and $\varepsilon > 0$. Selecting $y \in Y$ such that

$$F_p(f)(x_1) \geq f(y) + ||f|_p\, p(x_1 - y) - \varepsilon,$$

one obtains:

$$\begin{aligned}
&F_p(f)(x_2) - F_p(f)(x_1) \leq \\
&\leq f(y) + ||f|_p\, p(x_2 - y) - (f(y) + ||f|_p p(x_1 - y) - \varepsilon) \\
&= ||f|_p[p(x_2 - y) - p(x_1 - y)] + \varepsilon \\
&\leq ||f|_p\, p(x_2 - x_1)
\end{aligned}$$

(since $p(x_2 - y) - p(x_1 - y) \leq p(x_2 - x_1) \Longleftrightarrow p(x_2 - y) = p(x_2 - x_1 + x_1 - y) \leq p(x_2 - x_1) + p(x_1 - y)$).

The number $\varepsilon > 0$ being arbitrarily chosen, it follows

$$F_p(f)(x_2) - F_p(f)(x_1) \leq ||f|_p\, p(x_2 - x_1),$$

for all $x_1, x_2 \in X$, and then $||F_p(f)|_p \leq ||f|_p$.

Analogously, we obtain $||G_p(f)|_p \le ||f|_p$, and consequently

$$||G_p(f)|_p = ||f|_p = ||F_p(f)|_p.$$

d) Let $g \in p$-SLip$X$ such that $g|_Y = f$, and $||\mathrm{g}|_p = ||f|_p$. For every $y \in Y$ and $x \in X$ we have

$$f(y) - g(x) = g(y) - g(x) \le ||g|_p \, p(y-x),$$

and then

$$f(y) - ||g|_p \, p(y-x) \le g(x).$$

Taking the supremum with respect to $y \in Y$ we obtain

$$G_p(f)(x) \le g(x),$$

for every $x \in X$.

Analogously,

$$g(x) - f(y) = g(x) - g(y) \le ||g|_p \, p(x-y) = ||f|_p \, p(x-y),$$

implies

$$g(x) \le f(y) + ||f|_p p(x-y)$$

and taking the infimum with respect to $y \in Y$, it follows

$$g(x) \le F_p(f)(x),$$

for every $x \in X$. □

Remarks. 1) In Proposition 4 the condition "*Y is a $(p,\overline{p})$-bounded set*" is not necessary.

Indeed, for a nonvoid set Y and $f \in p$-SLip$Y$, the functions $F_p(f)$ and $G_p(f)$ are well defined. For this, let $y_0 \in Y$ and $x \in X$. For every $y \in Y$,

$$\begin{aligned} f(y) + ||f|_p p(x-y) &= f(y_0) + ||f|_p p(x-y) - (f(y_0) - f(y)) \\ &\ge f(y_0) + ||f|_p [p(x-y) - p(y_0-y)] \\ &= f(y_0) - ||f|_p [p(y_0-y) - p(x-y)]. \end{aligned}$$

But $p(y_0-y) - p(x-y) \le p(y_0-x) = \overline{p}(x-y_0)$, and then the set $\{f(y) + ||f|_p p(x-y) : y \in Y\}$ is bounded below and there exists

$$F_p(f)(x) = \inf_{y \in Y}\{f(y) + ||f|_p p(x-y)\},$$

for every $x \in X$.

Analogously, the function

$$G_p(f)(x) = \sup_{y \in Y}\{f(y) - ||f|_p p(y-x)\}$$

is well defined for every $x \in X$.

2) A proposition similar to Proposition 4 is valid for the functions of cone $\bar{p}$-SLip$Y$.

3) For $f \in p$-SLip$Y$ ($f \in \bar{p}$-SLip$Y$) let

$$E_p(f,Y) = \{g \in p\text{-SLip}X : g|_Y = f \text{ and } ||g|_p = ||f|_p\},$$

and respectively

$$E_{\overline{p}}(f,Y)=\{h\in\bar{p}\text{-SLip}X : h|_Y=f \text{ and } ||h|_{\overline{p}}=||f|_{\overline{p}}\},$$

the sets of extensions preserving the asymmetric seminorms $||f|_p$ (respectively $||f|_p$). We have the following inclusions:

$$E_p(f,Y)\subset S^+(0,||f|_p)=\{g\in p\text{-SLip}X : \|g|_p=||f|_p\},$$
$$E_{\overline{p}}(f,Y)\subset S^-(0,||f|_p)=\{h\in\bar{p}\text{-SLip}X :\ ||h|_{\overline{p}}=||f|_{\overline{p}}\}.$$

4) By Theorem of McShane [6], for every $f\in p^s$-SLip$Y$ there exists $F\in p^s$-SLip$X$, such that

$$F|_Y=f \text{ and } ||F||=||f||,$$

where $||f||$ is defined by formulas (6) and $||F||$ analogously.

Denote by

$$E_{p^s}(f,Y)=\{F\in p^s\text{-SLip}X : F|_Y=f \text{ and } ||F||=||f||\},$$

the set of all extensions of $f\in p^s$-SLip$Y$ preserving the Lipschitz constant $||f||$.

By Proposition 1, if $f\in p$-SLip$Y$, then $f\in p^s$-SLip$Y$ and because $||f|_p\leq ||f||$, it follows that

$$F_p(f)(x)\leq F(f)(x),\quad x\in X,$$

where

$$F(f)(x)=\inf_{y\in Y}\{f(y)+||f||\,p^s(x-y)\}.$$

If $f\in\bar{p}$-SLip$Y$, then $-f\in p$-SLip$Y$ and

$$F_p(-f)(x)\leq F(-f)(x),x\in X.$$

But

$$\begin{aligned}F_p(-f)(x)&=\inf_{y\in Y}\{-f(y)+||-f|_p p(x-y)\}\\&=-\sup_{y\in Y}\{f(y)-||-f|_p p(x-y)\}\\&=-\sup_{y\in Y}\{f(y)-||-f|_{\overline{p}}\overline{p}(x-y)\}\\&=-G_{\overline{p}}(f)(x),\end{aligned}$$

for every $x\in X$.

Analogously, if $f\in p$-SLip$Y$, then $-f\in\bar{p}$-SLip$Y$, and

$$F_{\overline{p}}(-f)(x)=-G_p(f)(x),\quad\forall x\in X.$$

If $G(f)(x)=\sup_{y\in Y}\{f(y)-||-f||\,p^s(x-y)\}$, then

$$F(-f)(x)=-G(f)(x),\quad x\in X,$$

and then

$$-G_{\overline{p}}(f)(x)\leq -G(f)(x),$$

and

$$G(f)(x)\leq G_{\overline{p}}(f)(x),\quad x\in X.$$

It follows that, if $f\in p$-SLip$Y$, then

$$G(f)\leq G_p(f)\leq F_p(f)\leq F(f)\ \text{ on } X,$$

and for $f\in\bar{p}$-SLip$Y$,

$$G(f)\leq G_{\overline{p}}(f)\leq F_{\overline{p}}(f)\leq F(f)\ \text{ on } X.$$

5) Let Y be a subspace of asymmetric normed space (X,p) and let φ_0 a bounded linear functional $\left(\varphi_0\in Y_p^b\right)$. By Proposition 3.1 [4] there exists $\varphi\in X_p^b$ such that

$$\varphi|_Y=\varphi_0 \text{ and } ||\varphi|_p=||\varphi_0|_p.$$

Let

$$E_p(\varphi_0)=\{\varphi\in X_p^b:\varphi|_Y=\varphi_0 \text{ and } ||\varphi|_p=||\varphi_0|_p\}$$

be the set of all extensions of φ_0 which preserves the asymmetric norm $||\varphi_0|_p$.

In this case $Y_p^b\subset p$-SLip$_0Y$ and for every $\varphi\in E_p(\varphi_0)$ the following inequalities hold:

$$G_p(\varphi)(x)\leq\varphi(x)\leq F_p(\varphi)(x),\ \ x\in X.$$

If $\varphi_0\in Y_p^b$ but $\varphi_0\notin Y_{\overline{p}}^b$, then

$$\varphi_0\left(B^+(0,r)\right)=(-\infty,r||\varphi_0|_p),\ \ r>0.$$

If $\psi_0\in Y_{\overline{p}}^b$ but $\psi_0\notin Y_{p,}^b$, then

$$\psi_0\left(B^-(0,r)\right)=(-\infty,r||\psi_0|_{\overline{p}}),\ \ r>0.$$

If $\varphi\in Y_{\overline{p}}^b\cap Y_{p,}^b$, then

$$\varphi\left(B^+(0,r)\right)=(-r||\varphi|_{\overline{p}},r||\varphi|_p),\ \ r>0,$$
$$\varphi\left(B^-(0,r)\right)=(-r||\varphi|_p,r||\varphi|_{\overline{p}}),\ \ r>0.$$ □

EXAMPLE 1. [11] (see also [9]) Let $(\mathbb{R},u)$ be the asymmetric normed space with $u(a)=\max\{a,0\},a\in\mathbb{R}$. Then $\overline{u}(a)=\max\{-a,0\}$ and $u^s(a)=|a|,\ a\in\mathbb{R}$. *Let* $Y=[-1,2]$ and $f:Y\to\mathbb{R}$, $f(y)=4y-y^2$. Because $f(y_1)-f(y_2)\leq 6\max\{y_1-y_2,0\}$ for all $y_1,y_2\in[-1,2]$, it follows that $f\in\bar{u}$-SLip$Y$ and $||f|_u=6$. Then the functions

$$F_u(f)(x)=\begin{cases}-5, & x\in(-\infty,-1),\\ 4x-x^2, & x\in[-1,2],\\ 6x-8, & x\in(2,\infty)\end{cases}$$

and

$$G_u(f)(x)=\begin{cases}6x+1, & x\in(-\infty,-1),\\ 4x-x^2, & x\in[-1,2],\\ 4, & x\in(2,\infty)\end{cases}$$

are the maximal and minimal extensions of f on $(\mathbb{R},u)$ with the asymmetric norms

$$||F_u(f)|_u=||G_u(f)|_u=||f||_u=6.$$

If one considers the normed space $(\mathbb{R}, u^s)$, then the functions

$$F(f)(x) = \inf_{y\in[-1,2]}\{f(y) + 6|x-y|\}, \quad x \in \mathbb{R},$$
$$G(f)(x) = \sup_{y\in[-1,2]}\{f(y) + 6|x-y|\}, \quad x \in \mathbb{R}$$

are Lipschtiz extensions for f and

$$G(f)(x) \le G_u(f)(x) \le F_u(f)(x) \le F(f)(x), \quad x \in \mathbb{R}.$$

4. APPLICATION

Let (X, p) be an asymetric normed space, and p^s-$\mathrm{SLip}_0 X$ the normed space of all p^s-Lipschitz real functions on X, vanishing at $0 \in X$, with the norm defined by (6).

If Y is a subset of X with $0 \in Y$ let also the normed space p^s-$\mathrm{SLip}_0 Y$, with the norm $\|\cdot\|$.

By the McShane theorem [6] for every $f \in p^s$-$\mathrm{SLip}_0 Y$ there exists at least one function $F \in p^s$-$\mathrm{SLip}_0 X$ such that $F|_Y = f$ and $\|F\| = \|f\|$.

Let also

$$Y^\perp = \{G \in p^s\text{-}\mathrm{SLip}_0 X : G|_Y = 0\},$$

the annihilator of Y with respect to p^s-$\mathrm{SLip}_0 X$.

Consider the following *best approximation problem*: for $F \in p^s$-$\mathrm{SLip}_0 X$, find an element $G_0 \in Y^\perp$ such that

$$\|F - G_0\| = d(F, Y^\perp) = \inf\left\{\|F - G\| : G \in Y^\perp\right\}.$$

An element $G_0 \in Y^\perp$ such that the above infimum is attained is called best approximation element for F in $Y^\perp$. If every $F \in p^s$-$\mathrm{SLip}_0 X$ has at least a best approximation element, then $Y^\perp$ is called proximinal.

The following result appears in [7] (see also [8]).

PROPOSITION 5. *In the above notations, the following properties hold:*

(a)
$$d(F, Y^\perp) = \|F|_Y\|, \quad \forall F \in p^s\text{-}\mathrm{SLip}_0 X;$$

(b) *the set of all best approximation elements of F in $Y^\perp$ is $F - E_{p^s}(F|_Y, Y)$, where $E_{p^s}(F|_Y, Y) = \{H \in p^s\text{-}\mathrm{SLip}_0 X : H|_Y = F|_Y$ and $\|H\| = \|F|_Y\|\}$.*

Proof. (a) Let $F \in p^s$-$\mathrm{SLip}_0 Y$. Then for every $G \in Y^\perp$, $\|F|_Y\| = \|F|_Y - G|_Y\| \le \|F - G\|$, and taking the infimum with respect to $G \in Y^\perp$ one obtains

$$\|F|_Y\| \le d(F, Y^\perp).$$

For $F|_Y$, by the McShane theorem [6], there exists $H \in E_{p^s}(F|_Y, Y)$ such that $F|_Y = H|_Y$ and $\|H\| = \|F|_Y\|$. Then

$$\|F|_Y\| = \|F - (F - H)\| \ge d(F, Y^\perp).$$

(b) Obviously, for every best approximation element G_0 of F in $Y^\perp$,

$$(F - G_0)|_Y = F|_Y \text{ and } \|F - G_0\| = \|F|_Y\|.$$

It follows that $F - G_0 \in F - E_{p^s}(F|_Y, Y)$, and so $G_0 \in F - E_{p^s}(F|_Y, Y)$.

Let now $G_0 \in F - E_{p^s}(F|_Y, Y)$. Then there exists $H \in E_{p^s}(F|_Y, Y)$ such that $G_0 = F - H$. But then

$$\|F - G_0\| = \|H\| = \|F|_Y\| = d(F, Y^\perp). \qquad \square$$

Remark 2. If $F \in \bar{p}$-$\mathrm{SLip}_0 X$ then $\|F\| = \|F|_p$ and $F|_Y \in p$-$\mathrm{SLip}_0 Y$, $\|F|_Y\| = \|F|_Y|_p = \| - F|_Y|_{\bar{p}}$.

It then follows

$$d(F, Y^\perp) = d(-F, Y^\perp) = \|F|_Y|_p, \quad \forall F \in p\text{-SLip}_0 X. \qquad \square$$

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