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SINGULAR INTEGRAL OPERATORS. THE CASE OF AN UNLIMITED CONTOUR

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Abstract. Let Γ be a closed or unclosed unlimited contour, a shift $\alpha(t)$ maps homeomorphicly the contour Γ onto itself with preserving or reversing the direction on Γ and also satisfies the conditions: for some natural $n \geq 2$, $\alpha_n(t) \equiv t$, and $\alpha_j(t) \not\equiv t$ for $1 \leq j < n$. In this work we study subalgebra Σ of algebra $L(L_p(\Gamma, \rho))$, which contains all operators of the form

$$(M\varphi)(t) = \sum_{k=0}^{n-1} \left\{ a_k(t)\varphi(\alpha_k(t)) + \tfrac{b_k(t)}{\pi \mathrm{i}} \int_\Gamma \tfrac{\varphi(\tau)}{\tau - \alpha_k(t)} d\tau \right\}$$

with piecewise-continuous coefficients. The existence of such an isomorphism between Σ and some algebra $\mathfrak{A}$ of singular operators with Cauchy kernel that an arbitrary operator from Σ and its image are Noetherian or not Noetherian simultaneously is proved. It allows to introduce the concept of a symbol for all operators from Σ and, using the known results for algebra $\mathfrak{A}$, in terms of a symbol to receive conditions of Noetherian property.

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1. INTRODUCTION

Let Γ be a closed or unclosed unlimited contour, a shift $\alpha(t)$ maps homeomorphicly the contour Γ onto itself with preserving or reversing the direction on Γ and also satisfies the conditions: for some natural[1] $n \geq 2$, $\alpha_n(t) \equiv t$, and for $1 \leq j < n$, $\alpha_j(t) \not\equiv t$ $(t \in \Gamma; \alpha_j(t) = \alpha[\alpha_{j-1}(t)], j = 1, 2, \ldots, n-1, \alpha_0(t) \equiv t)$

$$\frac{\alpha_i'(t)(t-z_0)^2}{(\alpha_i(t)-z_0)^2} \in H(\Gamma) \quad (z_0 \in \mathbb{C}\backslash\Gamma). \tag{1.1}$$

The class of such functions we shall designate by $V(\Gamma)$. Obviously, $V(\Gamma)$ does not depend on the choice of a point $z_0 \in \mathbb{C}\backslash\Gamma$.

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[1]If α reverses the orientation on Γ, n is always equal to two.

A singular integral operator with a Carleman shift is defined to be the operator of the form

$$(1.2)\qquad \left(M\varphi\right)(t)=\sum_{k=0}^{n-1}\left\{a_k(t)\varphi(\alpha_k(t))+\tfrac{b_k(t)}{\pi \mathrm{i}}\int_{\Gamma}\tfrac{\varphi(\tau)}{\tau-\alpha_k(t)}\mathrm{d}\tau\right\},$$

where $a_k(t), b_k(t)$ are functions given on the contour Γ. For a limited Lyapunov contour Γ the Noether theory of operators of the form (1.2), build in [10, 22], and the algebra, generated by operators of the form (1.2), are considered in papers [2, 3] and others. In the mentioned works the complete continuity of operators $T_k = W^k S W^{-k} - S$, where $(W\varphi)(t) = \varphi(\alpha(t))$ and S is the operator of singular integration along Γ

$$\left(S\varphi\right)(t)=\tfrac{1}{\pi \mathrm{i}}\int_{\Gamma}\tfrac{\varphi(\tau)}{\tau-t}\mathrm{d}t\quad (t\in\Gamma),$$

was essentially used.

The situation is absolutely different if Γ is an unlimited contour. In this case [6, 21] operator W is unbounded, generally speaking, in spaces L_p; instead of the operator W it is necessary to consider [13]–[16] the operator

$$(1.3)\qquad (V\varphi)(t)=\left(\tfrac{\alpha(t)-z_0}{t-z_0}\right)^{\lambda}\varphi(\alpha(t))\quad (1/p<\lambda<1+1/p)$$

and, as was shown in [16], for $\lambda \neq 1$ the operators $V^k S V^{-k} - S$ are not completely continuous. The difficulties of the research of operators (1.2) along an unlimited contour consist in these facts.

In the present paper on the base of results of works [13]–[17] the least subalgebra Σ of the algebra $L(L_p(\Gamma,\rho))$, containing all operators of the form (1.2) with piecewise continuous containing all operators of the form (1.2) with piecewise continuous containing all operators of the form (1.2) with piecewise continuous coefficients, is studied. It is necessary to consider separately the case, when α preserves the orientation on Γ, and the case, when α reverses the orientation. The algebra Σ contains the set Σ_0 of all sums of compositions of operators of the form (1.2), and also operators, which are limits (in the sense of convergence by the norm of operators) of a sequence of operators from Σ_0 The research of the set Σ_0 is based on the suggested by I. Gohberg and N.Y. Krupnik [4] method of the study of "complicated" operators, which allows to receive necessary and sufficient conditions of Noetherian property of operators from Σ. In the paper the existence of such an isomorphism between Σ and some algebra $\mathfrak{A}$ of singular integral operators with a Cauchy kernel that an arbitrary operator from Σ and its image are simultaneously Noetherian or not Noetherian is proved. It allows to introduce the concept of a symbol for all operators from Σ and, using known results for algebra $\mathfrak{A}$ (see [4]), in terms of a symbol to receive conditions of Noetherian property for all operators from Σ, including for $\Sigma\backslash\Sigma_0$. Through the symbol the index of operators $A \in \Sigma$ can be also expressed.

The set of values of the determinant of a symbol $A(t,\mu)$ represents a closed continuous curve, which can be oriented in a natural way. The index of this curve (i.e. the number of turns about the origin), taken with the opposite sign, is equal to the index of the operator A.

2. SYMBOL OF SINGULAR INTEGRAL OPERATORS WITHOUT A SHIFT ALONG AN UNLIMITED CONTOUR

Let Γ be an unlimited contour. We shall name Γ admissible, if the contour $\tilde{\Gamma} = \gamma(\Gamma), \gamma(t) = (t - z_0)^{-1} \quad (z_0 \notin \Gamma)$ is a simple closed or unclosed Lyapunov contour. Let $\mathfrak{A}$ be the algebra, generated by singular integral operators of the form

$$A = cI + dS, \tag{2.1}$$

where $c(t), d(t)$ are piecewise continuous matrix-functions of the order $\mathbf{n}$ and $\mathbf{S}$ is a matrix operator of singular integration along Γ. The used by us methods of the research of operators from algebra Σ assume the use of the Noether theory of operators from $\mathfrak{A}$. In this connection, in this section we shall establish some results and formulate them so that it would be possible to use them conveniently in the case of an unlimited contour.

Let $t_1, t_2, \ldots, t_{m-1}$ be some various points on the unlimited curve Γ, $t_m = \infty$; $\rho, \beta_1, \beta_2 \ldots, \beta_{m-1}, \beta_m = \sum_{k=1}^{m-1} \beta_k + p - 2$ be real numbers, satisfying the relations: $-1 < \beta_k < p - 1, \quad k = 1, 2, \ldots, m$, and

$$\rho(t) = \prod_{k=1}^{m-1} |t - t_k|^{\beta_k}\,. \tag{2.2}$$

Let introduce the following notations: $L_p(\Gamma, \rho)$ is the Banach space L_p on the contour Γ with weight $\rho(t)$; $L_p^n(\Gamma, \rho)$ is the Banach space of $\mathbf{n}$-dimensional vector functions $f = \{f_i\}_{j=1}^n$ with components $f_i \in L_p(\Gamma, \rho), \quad \Lambda_n(\Gamma)$ is the set of all matrix-functions F(t) of the order $\mathbf{n}$, continuous at each point of the contour Γ, except, possibly, a finite number of points, at which these functions are continuous from the left and have finite limits from the right. For further it is convenient to introduce the operators $P = (I+S)/2$ and $Q = I - P$. Then a usual singular operator $A = cI + dS$ in the space $L_p^n(\Gamma, \rho)$ can be represented as $A = aP + bQ$, where $a = c + d$ and $b = c - d$ $(a, b \in \Lambda_n(\Gamma))$.

To the operator A its symbol will be assigned (see [4, 8]). Let introduce some notations, necessary for the definition of a symbol. Let us assume for the beginning that the contour is closed. By $\theta = \theta(t), f(t,\mu)$ and $h(t,\mu), t \in \overline{\Gamma}, (0 \le \mu \le 1)$ let us designate the following functions:

$$\theta(t) = \begin{cases} \pi - 2\pi(1 + \beta_k)/p, & \text{if } t = t_k (k = 1, 2, \ldots, m), \\ \pi - 2\pi/p, & \text{if } t \in \overline{\Gamma} \backslash \{t_1, t_2, \ldots, t_m\}. \end{cases}$$

$$f(t,\mu) = \begin{cases} \frac{\sin\theta\mu \exp(\mathrm{i}\theta\mu)}{\sin\theta \exp(\mathrm{i}\theta)}, & \text{if } \theta \ne 0, \\ \mu, & \text{if } \theta = 0 \end{cases}$$

and $h(t,\mu)=\sqrt{f(t,\mu)(1-f(t,\mu))}$, respectively.

The symbol of an operator A is defined to be a matrix-function $A(t,\mu)(t\in\overline{\Gamma},0\le\mu\le 1)$ of order $\mathbf{2n}$, defined by the equality

$$A(t,\mu)= \\ =\begin{pmatrix} f(t,\mu)a(t+0)+(1-f(t,\mu))a(t) & h(t,\mu)(b(t+0)-b(t)) \\ h(t,\mu)(a(t+0)-a(t)) & f(t,\mu)b(t)+(1-f(t,\mu))b(t+0)\end{pmatrix}. \tag{2.3}$$

If Γ is unclosed, we consider that t_1 is its beginning and $t_m=\infty$ is its end. If $a\in\Lambda_n(\Gamma)$, then we set

$$a(t_1)=\lim_{\Gamma\ni t\to t_1}a(t_1)\text{ and }a(\infty)=\lim_{\Gamma\ni t\to\infty}a(t).$$

At all points $t\in\Gamma\backslash\{t_1,\infty\}$ the symbol $A(t,\mu)$ of the operator $A=aP+bQ$ will be defined by the equality (2.3). At points t_1 and $t=\infty$ we set

$$A(t_1,\mu)= \\ =\begin{pmatrix} f(t_1,\mu)a(t_1)+(1-f(t_1,\mu)) & h(t_1,\mu)(b(t_1)-1) \\ h(t_1,\mu)(a(t_1)-1) & f(t_1,\mu)+(1-f(t_1,\mu))b(t_1)\end{pmatrix}, \tag{2.4}$$

$$A(\infty,\mu)= \\ =\begin{pmatrix} f(\infty,\mu)+(1-f(\infty,\mu))a(\infty) & h(\infty,\mu)(1-b(\infty)) \\ h(\infty,\mu)(1-a(\infty)) & f(\infty,\mu)b(\infty)+(1-f(\infty,\mu))\end{pmatrix}, \tag{2.5}$$

where, remind, in the definition of functions f and h with $t=t_m=\infty$ the number $\beta_m=\sum_{k=1}^{m-1}\beta_k$. The symbol $R(t,\mu)$ of any operator $R\in\mathfrak{A}$ is constructed from symbols of the operators $A=aP+bQ$.

This symbol will be written in the form

$$R(t,\mu)=\begin{pmatrix} a_{11}(t,\mu) & a_{12}(t,\mu) \\ a_{21}(t,\mu) & a_{22}(t,\mu)\end{pmatrix}, \tag{2.6}$$

where $a_{ij}(t,\mu)$ are blocks of the dimension $\mathbf{n}$.

Theorem 2.1. *In order that the operator $R(\in\mathfrak{A})$ be Noetherian in the space $L_p^n(\Gamma,\rho)$ it is necessary and sufficient that the condition:*

$$\det R(t,\mu)\neq 0\quad (t\in\overline{\Gamma},0\le\mu\le 1) \tag{2.7}$$

be satisfied. If condition (2.7) *is satisfied, then*

$$\operatorname{Ind}A=-\frac{1}{2\pi}\left\{\arg\frac{\det R(t,\mu)}{\det a_{22}(t,0)\det(a_{22}(t,1))}\right\}_{\substack{t\in\Gamma\\ 0\le\mu\le 1}}. \tag{2.8}$$

The number $\frac{1}{2\pi}\{\arg g(t,\mu)\}_{(t,\mu)\in\Gamma\times[0,1]}$ in the right-hand side of the equality (2.8) corresponds to the number of turns counter-clockwise of the curve $g(t,\mu)$ about the point $z=0$ in the complex plane.

The symbol of singular integrated operators can get a more perfect form in the case, when coefficients of operators are Noetherian. Really, if Γ is closed, then the symbol of the operator $A = aP + bQ$ is determined by the equality

$$A(t,\mu) = \begin{pmatrix} a(t) & 0 \\ 0 & b(t) \end{pmatrix}$$

Let Γ be an unclosed unlimited arc. As weight we shall take the function $\rho(t) = |t - t_1|^\beta$. Let us designate by $\Lambda = \Lambda(\Gamma)$ the spatial closed unlimited curve ($\Lambda \subset \overline{R^3}$), consisting of all points (x, y, z), satisfying the relations $x + \mathrm{i}y \in \overline{\Gamma}$, $-1 \le z \le 1$, $(1 - z^2)(x + \mathrm{i}y - t_1) = 0$.

By other words, the curve $\Lambda(\Gamma)$ consists of two copies of the curve Γ, located in the planes $z = 1$ and $z = -1$, and a straight line segment, parallel to the axis z and passing through the beginning of the unclosed contour Γ.

The contour Γ is oriented so that in the plane $z = 1$ the direction along $\Lambda(\Gamma)$ coincides with the direction along Γ (i.e. from the point t_1), and in the plane $z = -1$ is opposite. For the space $L_p(\Gamma, |t - t_1|^\beta) \quad (-1 < \beta < p - 1)$ we shall designate by $\Omega_{p,\rho}$ the function defined on $\Lambda(\Gamma)$ by the following equalities (see [8]):

$$\Omega_{p,\rho}(t,z) = \begin{cases} \frac{z(1+d^2) - \mathrm{i}\varepsilon(1-z^2)d}{1+z^2d^2}, & \text{for } t = t_1, \infty, \\ z, & \text{for } t \in \Gamma \backslash \{t_1, \infty\}, \end{cases}$$

where $d = \mathrm{ctg}\pi(1 + \beta)/p$, $\varepsilon = 1$ for $t = t_1$ and $\varepsilon = -1$ for $t = \infty$. The symbol of the operator $A = aI + bS$ will be defined to be the following matrix-function

$$A(t,z) = a(t) + \Omega_{p,\rho}(t,z)b(t). \tag{2.9}$$

Theorem 2.2. *Let a and b be continuous matrix-functions on $\overline{\Gamma}$ of the order* $\mathbf{n}$. *In order that the operator $A = aP + bQ$ be Noetherian in the space $L_p^n(\Gamma, \rho)$ it is necessary and sufficient that the condition :*

$$\det(a(t) + \Omega_{p,\rho}(t,z)b(t) \neq 0) \quad (t, z \in \Lambda(\Gamma))$$

be satisfied. If this condition is satisfied, then

$$\mathtt{Ind}\mathbf{A} = -\mathtt{ind}\det(a(t) + \Omega_{p,\rho}(t,z)b(t)).$$

The last formulas are in some sense more obvious and easier to check.

Let us consider the following example. Let $\Gamma = [0, \infty)$ and $\rho(x) = x^\beta$ $(-1 < \beta < p - 1)$. Let's set $z = \frac{\mathrm{e}^{2\pi\xi} - 1}{\mathrm{e}^{2\pi\xi} + 1} \quad (\xi \in \mathbb{R})$ assuming it to be continued on $\overline{\mathbb{R}}$ by the continuity. Then (see [18, 9, 19]) the symbol of an operator A can be represented as

$$A(x,\xi) = a(x) + b(x)\psi(x,\xi),$$

where

$$\psi(x,\xi) = \begin{cases} \frac{\mathrm{e}^{2\pi(\xi + \mathrm{i}\gamma)} + 1}{\mathrm{e}^{2\pi(\xi+\mathrm{i}\gamma)} - 1}, & \text{for } x = 0, \infty, \\ \frac{\mathrm{e}^{2\pi\xi} - 1}{\mathrm{e}^{2\pi\xi} + 1}, & \text{for } 0 \le x \le \infty, \end{cases}$$

and $\gamma = \frac{1+\beta}{p}$. Note that the domain of the symbol $A(x,\xi)$ is the boundary $\mathbb{X}$ of the set $[0,\infty]\times[-\infty,\infty]$.

The following proposition plays an important role further on.

THEOREM 2.3. *Let δ_k be some real numbers and $h(t) = \prod_{k=1}^{m-1}(t-t_k)^{\delta_k}$. If*

$$-\tfrac{1+\beta_k}{p} < \delta_k < 1 - \tfrac{1+\beta_k}{p},$$

$$-\frac{1+\sum_{k=1}^{m-1}\beta_k}{p} < \delta_m = -\sum_{k=1}^{m-1}\delta_k < 1 - \frac{1+\sum_{k=1}^{m-1}\beta_k}{p} \tag{2.10}$$

then the operator $H = h(t)Sh^{-1}(t)I$ belongs to the algebra $\mathfrak{A}$ and its symbol $H(t,\mu)$ has the form

$$H(t,\mu) = \begin{pmatrix} E_n & U(t,\mu)E_n \\ 0 & -E_n \end{pmatrix}, \tag{2.11}$$

where E_n is a unity matrix of dimension **n** *and*

$$U(t,\mu) = \begin{cases} \frac{4ih(t_k,\mu)\sin(\pi\delta_k)\exp(\pi i\delta_k)}{2if(t_k,\mu)\sin(\pi\delta_k)\exp(\pi i\delta_k)+1}, & \textit{for } t = t_k, k = 1,2,\dots,m, \\ 0, & \textit{for } t \in \Gamma\backslash\{t_1,\dots,t_m\}. \end{cases}$$

3. THE CONDITION OF NOETHERIAN PROPERTY OF OPERATORS FROM Σ. THE CASE OF A CLOSED CONTOUR

In this section we will construct such a homeomorphism Δ of algebra Σ to algebra $\mathfrak{A}$ that the operators $A \in \Sigma$ and $\Delta(A) \in \mathfrak{A}$ are simultaneously Noetherian or not Noetherian with the condition of Noether $\mathrm{Ind}A = \frac{1}{n}\mathrm{Ind}\Delta(A)$. The symbol of operator A is defined as symbol of respective operator $\Delta(A) \in \mathfrak{A}$. We shall establish that an operator A is Noetherian if and only if the determinant of its symbol is not equal to zero. The index of the operator A can be also expressed through its symbol.

Let the shift $\alpha(t)$ satisfy the conditions (0.1) and $\alpha_n(t) \equiv t$. We shall designate $t_n = \infty$ and we shall calculate the iteration $t_j = \alpha_j(t_n)(j = 1,2,\dots,n-1)$. Let's introduce the space $L_p(\Gamma,\rho)$ with weight

$$\rho(t) = \prod_{j=1}^{n-1}|t-t_j|^{\frac{P_\lambda-2}{n}}, \tag{3.1}$$

where λ is some real number, satisfying the condition

$$\tfrac{n-2}{p(n-1)} < \lambda < \tfrac{n(p+1)-2}{p(n+1)}. \tag{3.2}$$

It is obvious that $-1 < \frac{p_\lambda-2}{n} < p-1$ and $-1 < \frac{(p\lambda-2)(n-1)}{n} < p-1$.

The suppositions made about the weight $\rho(t)$ ensure (see [20, 7]) the boundedness of the operators S and

$$(V\varphi)(t)=\left(\tfrac{\alpha(t)-z_0}{t-z_0}\right)^{\lambda}\varphi(\alpha(t))$$

in the space $L_p(\Gamma,\rho)$. Let's designate $(A_k\varphi)(t)=\tilde{a_k}(t)\varphi(t)+\tilde{b_k}(t)(S\varphi)(t)$, where

$$\tilde{a_k}(t)=\left(\tfrac{\alpha_{n-k}(t)-z_0}{t-z_0}\right)^{\lambda}a_k(\alpha_{n-k}(t)),$$

$$\tilde{b_k}(t)=\left(\tfrac{\alpha_{n-k}(t)-z_0}{t-z_0}\right)^{\lambda}b_k(\alpha_{n-k}(t)),$$

then the operator M, defined by equality (1.2), takes the form

$$M=A_0+VA_1+\cdots+V^{n-1}A_{n-1}. \tag{3.3}$$

Everywhere we shall consider that the functions $\tilde{a_k}(t)$ and $\tilde{b_k}(t)\in\Lambda_1(\Gamma)$. The case when coefficients $\tilde{a_k},\tilde{b_k}\in\Lambda_n(\Gamma)$ can be absolutely similarly investigated. The following theorem thus is valid [16].

Theorem 3.1. *The operator M is bounded in the space $L_p(\Gamma,\rho)$.*

We pass to the construction of the homomorphism Δ, specified above. Let's designate by $\mathfrak{B}^n_{p,\rho}(\Gamma)$ the Banach algebra of all operators, which operate in the space $L^n_p(\Gamma,\rho)$, and by $\mathfrak{t}^n$ the two-sided ideal of algebra $\mathfrak{B}^n_{p,\rho}(\Gamma)$, which consists of all completely continuous operators.

Let's take an arbitrary point τ_0 on the contour Γ. Let's calculate the iterations $\tau_k=\alpha_k(\tau_0), k=1,2,\dots,n-1$. As was established in [10], we can assume points τ_k ordered in positive direction on Γ. The contour Γ is divided into $\mathbf{n}$ not intersecting arcs: $(\tau_0,\tau_1),(\tau_1,\tau_2),\dots,(\tau_{n-2},\tau_{n-1}),(\tau_{n-1},\tau_0,)$. Let's define the function $\omega(t)$ on (τ_0,τ_1) as follows: the function $\omega(t)$ is equal to zero outside the arc $[\tau_0,\tau_1]$ and $\omega(\tau_0)=1,\omega(\tau_1)=\varepsilon$ $(\varepsilon=e^{\frac{2\pi\mathrm{i}}{n}})$; on $[\tau_0,\tau_1]$, $\omega(t)$ is continuous, bounded and different from zero. By U(t) we shall designate the function

$$U(t)=\sum_{k=0}^{n-1}\varepsilon^k\omega(\alpha_{n-k}(t))\quad(t\in\Gamma).$$

It is clear that the function U(t) everywhere on Γ is continuous, bounded and different from zero, and everywhere on Γ it satisfies the condition $U(t)-\varepsilon^{-1}U(\alpha(t))=0$.

Let δ_{mj} be the Kronecker delta. Let's consider the operators

$$R=\left(\delta_{mj}U^{m-1}I\right)^n_{m,j=1},\qquad N=\left(\varepsilon^{(m-1)(j-1)}V^{m-1}\right)^n_{m,j=1},$$

which belong to the algebra $\mathfrak{B}^n_{p,\rho}(\Gamma)$ (here and below m is the row number, j is the column number). The operators R and N are invertible, and

$$R^{-1}=\left(\delta_{mj}U^{1-m}I\right)^n_{m,j=1},\qquad N^{-1}=\tfrac{1}{n}\left(\varepsilon^{(1-m)(j-1)}V^{1-j}\right)^n_{m,j=1}.$$

Let's define the homeomorphism $\delta : \mathfrak{B}_{p,\rho}(\Gamma) \to \mathfrak{B}^n_{p,\rho}(\Gamma)$ as follows. If $A \in \mathfrak{B}_{p,\rho}(\Gamma)$, then we assume

$$\delta(A) = NR\Theta(A)R^{-1}N^{-1}, \tag{3.4}$$

where $\Theta(A) = (\delta_{mj}A)^n_{m,j=1} \in \mathfrak{B}^n_{p,\rho}(\Gamma)$.

LEMMA 3.1. *The operators A and $\delta(A)$ are Noetherian only simultaneously, and with the condition of Noether*

$$\mathtt{Ind}A = \frac{1}{n}\mathtt{Ind}\delta(A).$$

For proof see [6].

THEOREM 3.2. *The contraction Δ of homeomorphism δ to algebra Σ is also a homeomorphism*

$$\Delta : \Sigma \to \mathfrak{A},$$

and, if M is the operator (3.3), *then*[2]

$$\Delta(M) \simeq \tilde{M}, \tag{3.5}$$

where $\tilde{M} = \beta(t)I + \gamma(t)\tilde{S}$, $\beta(t) = (\,\tilde{a}_{-r+k+n}(\alpha_{k-1}(t)))^n_{r,k=1}$, $\gamma(t) = \left(\,\tilde{b}_{-r+k+n}(\alpha_{k-1}(t))\right)^n_{r,k=1}$ *and $\tilde{S}$ is defined in $L^n_p(\Gamma,\rho)$ by the equality*

$$\tilde{S} = \begin{pmatrix} S & & & 0 \\ & (t-t_1)^{1-\lambda}S(t-t_1)^{\lambda-1} & & \\ & & \ddots & \\ 0 & & & (t-t_{n-1})^{1-\lambda}S(t-t_{n-1})^{\lambda-1} \end{pmatrix}.$$

Proof. It is sufficient to find out the way in which the homomorphism Δ acts on operators of the form (3.3). Let $M_j = \sum_{r=0}^{n-1} \varepsilon^{rj} V^r A_r$ $(j = 0, 1, \dots, n-1)$, $A_r = \tilde{a}_r I + \tilde{b}_r S$, $L = (\delta_{mj}M_{j-1})^n_{m,j=1}$ $(M_0 = M)$ and $\Phi = \left(V^{k-1}A_{-m+k+n}V^{1-k}\right)^n_{m,k=1}$ $(A_{n+i} = A_i)$

Then, as is known, the following equality is satisfied:

$$L = N^{-1}\Phi N. \tag{3.6}$$

The immediate check shows that $U^j M \simeq M_j U^j$. Then

$$L \simeq R\Theta(M)R^{-1}. \tag{3.7}$$

As (see [16]) $V^r S V^{-r} \simeq (t-t_r)^{1-\lambda}S(t-t_r)^{\lambda-1}$ $(r = 1, \dots, m-1)$, then $\Phi \simeq \tilde{M}$. From here it follows that

$$\Delta(M) \simeq \tilde{M}.$$

The theorem is proved. $\square$

[2]Here and further by $K_1 \simeq K_2$ such operators that the operator $K_1 - K_2$ is completely continuous are designated.

The following theorem results from Lemma 3.1 and Theorem 3.2:

THEOREM 3.3. *If A is an arbitrary operator from algebra Σ, then operators A and $\Delta(A)$ are Noetherian or not Noetherian simultaneously, with Noether condition*

$$\mathtt{Ind}A = \frac{1}{n}\mathtt{Ind}\Delta(A).$$

As the operator $\Delta(A) \in \mathfrak{A}$ and for operators from algebra $\mathfrak{A}$ the concept of symbol was introduced, then a symbol of an operator $A \in \Sigma$ can be naturally defined to be the symbol of the operator $\tilde{A} = \Delta(A)$. Thus, the symbol of an operator $A \in \Sigma$ is the matrix-function $A(t,\mu) = \tilde{A}(t,\mu)$ of the order $2n$. In particular, the symbol of the operator $M = A_0 + VA_1 + \cdots + V^{n-1}A_{n-1}$ $(A_j = \tilde{a}I + \tilde{b}S)$ according to the formulas (2.3) and (2.10) has the form

$$M(t,\mu) =$$
$$= \begin{pmatrix} f(t,\mu)\beta(t+0) + (1 - f(t,\mu)\beta(t)) & h(t,\mu)(\beta(t+0) - \beta(t)) \\ h(t,\mu)(\beta(t+0) - \beta(t)) & f(t,\mu)\beta(t) + (1 - f(t,\mu)\beta(t+0)) \end{pmatrix}$$
$$+ \begin{pmatrix} f(t,\mu)\gamma(t+0) + (1 - f(t,\mu))\gamma(t) & (\gamma(t+0) - \gamma(t)) \\ h(t,\mu)(\gamma(t+0) - \gamma(t)) & f(t,\mu)\gamma(t) + (1 - f(t,\mu))\gamma(t+0) \end{pmatrix}$$
$$\cdot \tilde{S}(t,\mu),$$

where

$$\tilde{S}(t,\mu) = \left(\begin{array}{ccc|cccc}
1 & & 0 & 0 & & & 0 \\
 & & & & U_1(t,\mu) & & \\
 & \ddots & & & & \ddots & \\
0 & & 1 & 0 & & & U_{n-1}(t,\mu) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & -1 & & & 0 \\
 & 0 & & & & \ddots & \\
 & & & 0 & & & -1
\end{array} \right)$$

and functions $U_j(t,\mu)$, because of formula (2.11), are defined by the equalities

$$U_j(t,\mu) = \begin{cases} \frac{-4\mathrm{i}h(t_j,\mu)\sin(\pi\lambda)\exp(-\pi\mathrm{i}\lambda)}{-2\mathrm{i}f(t_j,\mu)\sin(\pi\lambda)\exp(-\pi\mathrm{i}\lambda)+1}, & \text{if } t = t_j; \\ \frac{4\mathrm{i}h(t_n,\mu)\sin(\pi\lambda)\exp(\pi\mathrm{i}\lambda)}{2\mathrm{i}f(t_n,\mu)\sin(\pi\lambda)\exp(\pi\mathrm{i}\lambda)+1}, & \text{if } t = t_n; \\ 0, & \text{if } t \in \Gamma\backslash\{t_1,\dots,t_n\}. \end{cases}$$

If

$$M = \sum_{j=1}^{m} M_{j1}M_{j2}\dots M_{jr}, \tag{3.8}$$

where M_{ji} are operators of the form (3.3), then

$$M(t,\mu) = \sum_{j=1}^{m} M_{j1}(t,\mu) M_{j2}(t,\mu) \dots M_{jr}(t,\mu), \tag{3.9}$$

where $M_{ji}(t,\mu)$ is the symbol of operator M_{ji}. The set of all operators of the form (3.8) we shall designate by Σ_0. By the method described in [4] one can prove the following theorem:

THEOREM 3.4. *If the operator M belongs to algebra Σ_0 and $M(t,\mu) = \|a_{jk}(t,\mu)\|_{j,k=1}^{2}$ is its symbol, then*

$$\max_{t\in\Gamma;0\le\mu\le1} |a_{jk}(t,\mu)| \le \inf_{T\in\mathfrak{F}} \|A+T\|, \tag{3.10}$$

where $\mathfrak{T}$ is the set of all completely continuous operators in the space $L_p(\Gamma,\rho)$.

COROLLARY 3.1. *The symbol of operator $M \in \Sigma_0$ does not depend on representation of an operator in the form*

$$M = \sum_{j=1}^{m} M_{j1} M_{j2} \dots M_{jr}.$$

COROLLARY 3.2. *The map $M \to M(t,\mu)$ of algebra Σ_0 to algebra of symbols is an algebraic homeomorphism, whose kernel contains the set of all completely continuous operators from algebra Σ_0.*

Let $A \in \Sigma\backslash\Sigma_0$ and $A = \lim_{n\to\infty} A_n$, where $A_n \in \Sigma_0$. By virtue of the relation (3.10), symbols of operators A_n are uniformly convergent to some continuous matrix-function $A(t,\mu)$, which does not depend on the choice of a sequence $\{A_n\}_1^\infty$, and which we shall name the symbol of the operator A.

From Theorem 3.3 and from a property of the symbol of algebra $\mathfrak{A}$ follows

THEOREM 3.5. *In order that the operator $A \in \Sigma$ be Noetherian in the space $L_p(\Gamma,\rho)$ it is necessary and sufficient that its symbol*

$$A(t,\mu) = \begin{pmatrix} a_{11}(t,\mu) & a_{12}(t,\mu) \\ a_{21}(t,\mu) & a_{22}(t,\mu) \end{pmatrix}$$

be non-degenerate: $\det A(t,\mu) \ne 0$ $(t \in \Gamma, 0 \le \mu \le 1)$. *If this condition is fulfilled, there exists a two-sided regulator of the operator,*[3] *belonging to algebra Σ and*

$$\mathtt{Ind}A = -\tfrac{1}{2n\pi}\Big\{\arg \frac{\det A(t,\mu)}{\det a_{22}(t,0)\det a_{22}(t,1)}\Big\}_{t\in\Gamma,\ 0\le\mu\le1}.$$

[3]If A is Noetherian, then there exists such a linear bounded operator B that operators $I - AB$ and $I - BA$ are completely continuous; B is referred to as a two-sided regulator of the operator A.

4. THE CONDITION OF NOETHERIAN PROPERTY OF OPERATORS FROM Σ. CASE OF AN UNCLOSED CONTOUR

I. Continuous coefficients.

Let's consider at first the case, when the algebra Σ is generated by operators S, V and operators of multiplication by continuous functions. Let Γ be an unclosed contour with the beginning $t = t_1$ and the extremity $t = \infty$ and $\alpha: \ \Gamma \to \Gamma$ satisfies the condition $\alpha(\alpha(t)) \equiv t$, then it is obvious that $\alpha(t_1) = \infty$ and $\alpha(\infty) = t_1$. Let

$$\rho(t) = |t - t_1|^\beta \ \left(\beta = \tfrac{p\lambda - 2}{2}, \ 1/_p < \lambda < 1 + 1/_p\right).$$

Lemma 4.1. *Each operator $A \in \Sigma$ can be represented as a sum $A = A_1 + A_2 V$ $(A_1, A_2 \in \mathfrak{A})$ and it is uniquely determined up to completely continuous summands.*

Proof. Obviously, it is enough to show that the set of operators of the form

$$A_1 + A_2 V, \tag{4.1}$$

where A_1 and A_2 run algebra $\mathfrak{A}$, is a Banach algebra and from the equality $A_1 + A_2 V = 0$, $(A_1, A_2 \in \mathfrak{A})$ it follows that A_1, A_2 are completely continuous operators. From the equality $VSV = -(t - t_0)^{1-\lambda} S (t - t_0)^{\lambda - 1} + T$ $(\in \mathfrak{A})$ and $VaV = a(\alpha(t))I$ it follows that the sum and the product of operators of the form (4.1) have the same form. Let's prove that this set is closed.

Let $A_1^{(n)} + A_2^{(n)} V$ converge uniformly to A. Let U be a invertible operator, built in [13], satisfying the condition : $U^{-1}(A_1^{(n)} + A_2^{(n)} V) U = A_1^{(n)} - A_2^{(n)} V + T_n$. Then $2\hat{A}_1^{(n)} \to \hat{A} + \hat{U}^{-1} \hat{A} \hat{U} = 2\hat{A}_1 \in \hat{\mathfrak{A}}$, where $\hat{\mathfrak{A}} = \mathfrak{A}/\mathfrak{T}(L_p(\Gamma, \rho))$ is the factor-algebra in all completely continuous operators, acting in the space $L_p(\Gamma, \rho)$. Analogously, $\hat{A}_2^{(n)} \to \hat{A}_2 \in \hat{\mathfrak{A}}$. So, $\hat{A} = \hat{A}_1 + \hat{A}_2 V$ and, hence, $A = A_1 + A_2 V$ $(A_1, A_2 \in \mathfrak{A})$.

Let $A_1 + A_2 V = 0$, then $\hat{A}_1 + \hat{A}_2 \hat{V} = 0$ and $\hat{A}_1 - \hat{A}_2 \hat{V} = \hat{U}^{-1}(\hat{A}_1 + \hat{A}_2 \hat{V})\hat{U}$, i.e. $A_1, A_2 \in \mathfrak{T}(L_p(\Gamma, \rho))$. $\square$

Lemma 4.2. *Let $A, B, W \in L(\mathfrak{B})$ and $W^2 = I$, then the equality is satisfied (see [10])*

$$\begin{pmatrix} I & W \\ I & -W \end{pmatrix} \begin{pmatrix} A & B \\ WBW & WAW \end{pmatrix} \begin{pmatrix} I & I \\ W & -W \end{pmatrix} = \\ = 2 \begin{pmatrix} A + BW & 0 \\ 0 & A - BW \end{pmatrix}. \tag{4.2}$$

The proof is checked immediately (see [1]).

Theorem 4.1. *Let $A, B \in \mathfrak{A}$ and $(V\phi)(t) = \left(\frac{\alpha(t) - z_0}{t - z_0}\right)^\lambda \phi(\alpha(t))$. The operator $R = A + BV$ is Noetherian in the space $L_p(\Gamma, \rho)$ if and only if the* *operator*

$$R_V = \begin{pmatrix} A & B \\ VBV & VAV \end{pmatrix} \tag{4.3}$$

is Noetherian in the space L_p^2. *If* $A + BV$ *is Noetherian, then*

$$\mathtt{Ind}(A + BV) = \tfrac{1}{2}\mathtt{Ind}\begin{pmatrix} A & B \\ VBV & VAV \end{pmatrix}. \tag{4.4}$$

Proof. Let's write the equality (4.2) for operators A, B, V. Remark that $U^{-1}(A - BV)U = A + BV + T$, hence, $A + BV$ is Noetherian only simultaneously with the operator $A - BV$ and $\mathtt{Ind}(A + BV) = \mathtt{Ind}(A - BV)$. Since extreme multiplicands in the left-hand side of equalities (4.2) are invertible operators (their product is $2I$), all the statements of theorem result from the equality (4.2). Theorem is proved. □

Remark 4.1. The matrix operator (4.3) is a singular operator with matrix coefficients (without shift!). □

Corollary 4.1. *Let* $A = aI + bS$, $B = cI + dS$, $a, b, c, d \in C(\overline{\Gamma})$, $R = A + BV$ *and* $(V\varphi)(t) = \frac{\alpha(t) - z_0}{t - z_0}\varphi(\alpha(t))$. *Then,*

$$R_V = \begin{pmatrix} A & B \\ VBV & VAV \end{pmatrix} = \begin{pmatrix} a & c \\ \tilde{c} & \tilde{(a)} \end{pmatrix} + \begin{pmatrix} b & d \\ -\tilde{d} & -\tilde{d} \end{pmatrix} S + T, \tag{4.5}$$

where $\tilde{f}(t) = f(\alpha(t))$ *and* T *is a completely continuous operator in* $L_p^2(\Gamma, \rho)$.

The symbol of the operator $R = A + BV$ will be defined to be the matrix $R_V(t, \mu)$ $t \in \overline{\Gamma}$, $0 \le \mu \le 1$, defined by equalities (2.3), (2.4) and (2.5). From above reasonings and Theorems 2.1 and 4.1 results

Theorem 4.2. *In order that the operator* $R = A + BV$ *be Noetherian in the space* $L_p(\Gamma, \rho)$ *it is necessary and sufficient that*

$$\det R_V(t, \mu) \neq 0 \quad (t \in \overline{\Gamma}, 0 \le \mu \le 1).$$

If this condition is satisfied, then

$$\mathtt{Ind}R = -\frac{1}{4\pi}\left\{\arg\frac{\det R_V(t, \mu)}{\det a_{22}(t, 0)\det a_{22}(t, 1)}\right\}_{t \in \Gamma,\ 0 \le \mu \le 1}.$$

Remark 4.2. The symbol of the operator R can be defined by the equality (2.9) if substitute for the matrices a and b the respective matrices of the operator R_V. In this case Theorem 2.2 for the operator $R = A + BV$ is true. □

II. Discontinuous coefficients.

Theorems 4.1 and 4.2 generally speaking can not be transferred to the case, when the coefficients $a, b, c, d \in \Lambda_1(\Gamma)$. To show this it is enough to give an example of two singular operators A and B such that $A+BV$ is Noetherian, but

$A-BV$ is not Noetherian. From equation (4.2) it results that the respective matrix operator is not Noetherian. The examples for $\Gamma=[-1,1]$ or $\Gamma=\mathbb{R}$ and $\alpha(x)=-x$ are built in works [15,1]. After constructing the symbol of operators of the form $A+BV$ we can construct such examples for every Γ and $\alpha\in V(\Gamma)$ (see §4).

Let introduce some notations. Let's designate by $\Sigma(\Gamma,\rho)$ the Banach subalgebra of the algebra $L(L_p(\Gamma,\rho)),(\rho(t)=|t-t_1^\beta|,\beta=\frac{p\lambda-2}{2})$, generated by an operator S, completely continuous operators and all operators of multiplication by function $a\in\Lambda_1(\Gamma)$. By $\Sigma(\Gamma,\rho;B)(B\in L(L_p(\Gamma,\rho)))$ we shall designate the Banach subalgebra generated by all operators from $\Sigma(\Gamma,\rho)$ and by the operator B. Let Σ be any subalgebra of algebra $L(\mathfrak{B})$. By Σ_n we shall designate the subalgebra of the algebra $L(\mathfrak{B}_n^n)$ consisting of all operators of the form $(A_{jk})_{j,k=1}^n$, where $A_{jk}\in\Sigma$.

Definition 4.1. *Two algebras $\mathfrak{A}_1(\subset L(\mathfrak{B}_1))$ and $\mathfrak{A}_2(\subset L(\mathfrak{B}_2))$ will be called equivalent* [1] *if there is an invertible operator $M\in L(\mathfrak{B}_1,\mathfrak{B}_2)$ such that the set of operators of the form $MAM^{-1}(A\in\mathfrak{A}_1)$ coincides with the algebra $\mathfrak{A}_2$.*

Theorem 4.3. *Let Γ be some closed or unclosed contour in the complex plane and $\rho(t)=\prod\limits_{k=1}^n|t-t_k|^{\beta_k},(-1<\beta_k<p-1)$, then the algebra $\Sigma(\Gamma,\rho)$ is equivalent with the algebra $\Sigma(\Gamma)=\Sigma(\Gamma,1)$.*

Proof. Let $(M\phi)(t)=\rho^{1/p}(t)\phi(t)$. As $\|M\phi\|_{L_\rho(\Gamma)}=\|\phi\|_{L_\rho(\Gamma,\rho)}$, M maps isometrically the space $L_p(\Gamma,\rho)$ on $L_p(\Gamma)$. As $MaM^{-1}=aI$ for any function $a\in\Lambda_1(\Gamma)$, therefore it is enough to show that $MSM^{-1}\in\Sigma(\Gamma)$. The operator MSM^{-1} can be represented as

$$MSM^{-1}=f(t)\prod_{k=1}^n(t-t_k)^{\beta_k/p}S\prod_{k=1}^n(t-t_k)^{-\beta_k/\rho}f^{-1}(t)I$$

where $f(t)$ is different from zero and continuous everywhere on Γ with exception, perhaps, the points t_k. The numbers $\alpha_k=\beta_k/p$ satisfy the condition

$$-1/p<\alpha_k<1-\tfrac{1}{p},$$

therefore (see [20]) the operator $\prod_{k=1}^n(t-t_k)^{\beta_k/p}S\prod_{k=1}^n(t-t_k)^{-\beta_k/p}I$ belongs to the algebra $\Sigma(\Gamma)$. Theorem is proved. □

Remark 4.3. If Γ is unlimited, it is necessary to require that numbers β_k together with the conditions $-1<\beta_k<p-1$, $(k=1,\dots,n)$ satisfy the condition $1<\sum\limits_{k=1}^n\beta_k<p-1$. □

Corollary 4.2. *Any two algebras $\Sigma(\Gamma,\rho_1)$ and $\Sigma(\Gamma,\rho_2)$ are equivalent.*

Further on we shall consider the algebra $\Sigma(\Gamma,\rho;V)$, where $\rho(t)=|t-t_1|^\beta$ ($\beta=\frac{p\lambda-2}{2}, 1/p<\lambda<1+1/p$) and

$$(V\phi)(t)=\Big(\tfrac{\alpha(t)-z_0}{t-z_0}\Big)^\lambda \quad (\alpha\in V(\Gamma)).$$

Remark that we can assume that $t_1=0$, as otherwise it can be achieved with the help of the invertible operator $H:L_p(\Gamma,\rho)\to L_p(\tilde{\Gamma},\tilde{\rho})$, defined by the equality $(H\phi)(z)=\phi(z+t_1)((H^{-1}\phi)(t)=\phi(t-t_1))$, where $\tilde{\rho}(t)=|z|^\beta$ and $\tilde{\Gamma}=\{z=t-t_1:t\in\Gamma\}$. For the same reason it is possible to consider that $z_0\notin\Gamma$. Besides it is enough to study the algebra $\Sigma(\mathbb{R}^+,x^\beta;W)$, where

$$(W\varphi)(x)=\Big(\tfrac{\alpha(x)+1}{x+1}\Big)^\lambda\phi(\alpha(x)),\ \ \alpha\in V(\mathbb{R}^+).$$

and $\alpha\in V(\mathbb{R})$. Really, let $\gamma:\mathbb{R}^+\to\Gamma$ be a homeomorphic map of $\mathbb{R}^+$ on Γ with the following properties: $\gamma(0)=1,\ \gamma(\infty)=\infty$ and $0\neq\gamma'(x)\in H(\mathbb{R}^+)$. Obviously (see [5]), such a function exists. The operator M, defined by the equality

$$(M\varphi)(x)=\varphi(\gamma(x)),$$

is a linear bounded invertible operator, acting from $L_p(\Gamma,|t|^\beta)$ into $L_p(\mathbb{R}^+,x^\beta)$. Let's designate by ω the function $\omega(t)=\gamma^{-1}(t)$. Then $(M^{-1}\phi)(t)=\phi(\omega(t))$, $MSM^{-1}=S_++T$, where

$$(S_+\varphi)(x)=\tfrac{1}{\pi\mathrm{i}}\int_0^\infty\tfrac{\varphi(y)\mathrm{d}y}{y-x},$$

$$(T\varphi)(x)=\tfrac{1}{\pi\mathrm{i}}\int_0^\infty\Big(\tfrac{\gamma'(y)}{\gamma(y)-\gamma(x)}-\tfrac{1}{y-x}\Big)\varphi(y)\mathrm{d}y$$

is completely continuous in $L_p(\mathbb{R}^+,x^\beta)$. It is easy to see that $MaM^{-1}=a(\gamma(x))I$ and $MVM^{-1}\varphi=(\frac{\alpha(\gamma(x))+1}{\gamma(x)+1})^\lambda\varphi((\omega\circ\alpha\circ\gamma)(x))$. Let's designate $\nu(x)=(\omega\circ\alpha\circ\gamma)(x)$. From properties of the functions $\gamma,\ \omega$ and α easily follows that $\nu\in V(\mathbb{R}^+)$. The operator MVM^{-1} can be represented as $MVM^{-1}=fW$, where $f(x)=(\frac{\alpha(\gamma(x))+1}{\gamma(x)+1})^\lambda(\frac{x+1}{\gamma(x)+1})^\lambda\in C(\overline{\mathbb{R}^+})$, is not equal to zero and $(W\varphi)(x)=(\frac{\nu(x)+1}{x+1})^\lambda\varphi(\nu(x))$. Thus $M\Sigma(\Gamma,|t|^\beta;V)M^{-1}=\Sigma(\mathbb{R}^+,x^\beta;W)$. So, further on we shall consider the algebra $\Sigma(\mathbb{R}^+,x^\beta;W)$ and we shall show that it is equivalent to some algebra generated by singular integral operators without a shift.

Theorem 4.4. *Let $\alpha\in V(\mathbb{R}^+)$,*

$$(W\varphi)(x)=\Big(\tfrac{\alpha(x)+1}{x+1}\Big)^\lambda\varphi(\alpha(x)), \tag{4.6}$$

$\rho(x)=x^\beta$ ($\beta=\frac{p\lambda-2}{2},\ 1/p<\lambda<1+1/p$). *The algebra $\Sigma(\mathbb{R}^+,\rho;W)$ is equivalent to the algebra $\Sigma(\Gamma_0,\rho_0,V_0)$, where $\Gamma_0=[-1,1]$, $\rho_0(t)=(1-t^2)^\beta$ and $(V_0\varphi)(t)=\varphi(-t)$.*

To prove this theorem, we need the following lemma.

Lemma 4.3. *Let $\alpha \in V(\mathbb{R}^+)$ and $\alpha(x_0) = x_0$, then there exists a homeomorphic map $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ with the following properties:*

$$\mu(0) = 0, \;\; \mu(x_0) = 1, 0 \neq \mu'(x) \in H(\mathbb{R}^+), \tag{4.7}$$

$$(\mu^{-1} \circ \alpha \circ \mu)(x) = \tfrac{1}{x}. \tag{4.8}$$

Proof. Let $\tilde{\mu}(x) = \frac{x}{x_0}$ ($\tilde{\mu}^{-1}(x) = x \cdot x_0$), then, obviously, the map $\nu = \tilde{\mu} \circ \alpha \circ \tilde{\mu}^{-1}$ belongs to the set $V(\mathbb{R}^+)$, and $\nu(1) = 1$. The map

$$\omega(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ \frac{1}{\nu(x)}, & \text{if } x \in [1,+\infty], \end{cases} \tag{4.9}$$

satisfies the following conditions. There exists

$$\omega^{-1}(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ \gamma(\frac{1}{x}), & \text{if } x \in [1,\infty], \end{cases} \tag{4.10}$$

where $\omega'(x) \in H([0,1])$, $\omega'(x) \in H([1,\infty])$, $\omega'(x) \neq 0$ and $\omega(1) = 1$. Let's show that $\omega'(x)$ is continuous at the point $x = 1$. This implies that $\omega'(x) \in H(\mathbb{R}^+)$. So, we have

$$\omega'(1-0) = 1, \quad \omega'(1+0) = \lim_{x \to 1} -\tfrac{\nu'(x)}{\nu^2(x)} = \tfrac{-\nu'(1)}{\nu^2(1)} = 1,$$

since ν reverses the orientation on $\mathbb{R}^+$ and $\nu'(1) = -1$. The function $\mu = \tilde{\mu}^{-1} \circ \omega^{-1}$ satisfies the conditions (4.7). Let show that $(\mu^{-1} \circ \alpha \circ \mu)(x) = 1/x$. We have $\mu^{-1} \circ \alpha \circ \mu = \omega \circ \tilde{\mu} \circ \alpha \circ \tilde{\mu}^{-1} \circ \omega^{-1} = \omega \circ \nu \circ \omega^{-1}$. Let $x \in [0,1]$, then $(\omega \circ \nu \circ \omega^{-1})(x) = \omega(\nu(\omega^{-1}(x))) = \omega(\nu(\nu(\frac{1}{x}))) = \omega(\frac{1}{x}) = \frac{1}{x}$. Lemma is proved. □

Corollary 4.3. *The algebra $\Sigma(\mathbb{R}^+, x^\beta; W)$ is equivalent to the algebra $\Sigma(\mathbb{R}^+, x^\beta; W_0)$, where*

$$(W_0\varphi)(x) = \tfrac{1}{x^\lambda}\varphi\left(\tfrac{1}{x}\right). \tag{4.11}$$

Indeed, let's designate by M_1 the operator $(M\varphi)(x) = \varphi(\mu(x))$, where μ is the function from Lemma 4.2. Then the operator $M_1 S_+ M_1^{-1} - S_+$ is an integral operator with the kernel $\frac{1}{\pi i}(\frac{\mu'(y)}{\mu(y)-\mu(x)} - \frac{1}{x-y})$, and, according to property (4.7), it is completely continuous in $L_p(\mathbb{R}^+, x^\beta)$. Besides, $M_1 a M_1^{-1} = a(\mu(x))I \in \Sigma(\mathbb{R}^+, x^\beta; W_0)$ for any function $a \in \Lambda_1(\mathbb{R}^+)$. Let's show that $M_1 W M_1^{-1} \in \Sigma(\mathbb{R}^+, x^\beta; W_0)$. So, we have

$$\begin{aligned}(M_1 W M_1^{-1}\varphi)(x) &= M_1 W \varphi(\mu^{-1}(x)) = M_1(\tfrac{\alpha(x)+1}{x+1})^\lambda \varphi(\mu^{-1}(\alpha(x))) = \\ &= (\tfrac{\alpha(\mu(x))+1}{\mu(x)+1})^\lambda \varphi\left(\tfrac{1}{x}\right) = f(x)(W_0\varphi)(x),\end{aligned}$$

where the function $f(x) = [\frac{(\alpha(\mu(x))+1)x}{\mu(x)+1}]^\lambda$ is continuous and different from zero on $\tilde{\mathbb{R}^+}$. This fact results from properties of functions μ and α (see [13]). Besides thereby, $M_1 W M_1^{-1} \in \Sigma(\mathbb{R}^+, x^\beta; W_0)$.

Proof of Theorem 4.4. Considering the above said, it remains to show that the algebra $\Sigma(\mathbb{R}^+, x^\beta; W_0)$ is equivalent with the algebra $\Sigma(\Gamma_0, \rho_0; V_0)$. Let M_2 be the operator acting by the rule:

$$(M_2\varphi)(t) = \tfrac{1}{1-t}\varphi\left(\tfrac{1+t}{1-t}\right). \tag{4.12}$$

The operator M_2 is linear, bounded and acting from $L_p(\mathbb{R}^+, x^\beta)$ into $L_p(\Gamma_0, h)$, where $\Gamma_0 = [-1, 1]$ and $h(t) = |1 + t|^\beta |1 - t|^{p-\beta-2}$. The operator M_2 is invertible, moreover,

$$(M_2^{-1}\Psi)(x) = \tfrac{2}{x+1}\Psi\left(\tfrac{x-1}{x+1}\right). \tag{4.13}$$

By simple calculations (we omit details) we make sure that $M_2S_+M_2^{-1} = S_0$, $M_2aM_2^{-1} = \tilde{a}I$ $(a \in \Lambda_1(\mathbb{R}^+))$ and $(M_2W_0M_2^{-1}\varphi)(t) = \left(\frac{1-t}{1+t}\right)^{\lambda-1}\varphi(-t)$ where $\hat{a}(t) = a\left(\frac{1+t}{1-t}\right) \in \Lambda_1(\Gamma_0)$ and $(S_0\varphi)(t) = \frac{1}{\pi \mathrm{i}}\int_{-1}^{1}\frac{\varphi(\tau)}{\tau-t}d\tau$ $(t \in \Gamma_0)$. Finally we'll consider the operator

$$(M_3\varphi)(t) = (1-t)^{1-\lambda}\varphi(t),$$

which isometrically maps the space $L_p(\Gamma_0, h)$ into the space $L_p(\Gamma_0, \rho_0)(\rho_0(t) = (1-t^2)^\beta)$. It is easy to see that $M_3\hat{a}M_3^{-1} = \hat{a}I$, $M_2S_0M_2^{-1} = (1-t)^{1-\lambda}S_0(1-t)^{\lambda-1}I$ and $M_3M_2W_0M_2^{-1}M_3^{-1} = V_0$, where $(V_0\varphi)(t) = \varphi(-t)$. Considering the condition $-1/p < 1-\lambda < 1-1/p$ and $\beta = \frac{p\lambda-2}{2}$, from Theorem 2.2 we conclude that the operator $(1-t)^{1-\lambda}S_0(1-t)^{\lambda-1}I$ belongs to the algebra $\Sigma(\Gamma_0, \rho_0)$. In particular its symbol can bee calculated with the help of (2.11). So, the algebra $\Sigma(\mathbb{R}^+, x^\beta; W_0)$ is equivalent to the algebra $\Sigma(\Gamma_0, \rho_0; V_0)$. Theorem is proved. □

Corollary 4.4. *The algebra* $\Sigma(\Gamma, \rho; V)$ *is equivalent with algebra* $\Sigma_2(\tilde{\Gamma}_0, \tilde{\rho}_0)$*, where* $\tilde{\rho}(t) = (1-t)^\beta t^{-1/2}$*,* $\tilde{\Gamma}_0 = [0, 1]$*.*

Really, from the proved above it follows that the algebra $\Sigma(\Gamma, \rho; V)$ is equivalent with the algebra $\Sigma(\Gamma_0, \rho_0; V_0)$. It remains to use results of work [1], in which it is proved that $\Sigma(\Gamma_0, \rho_0; V_0) \sim \Sigma(\tilde{\Gamma}_0, \tilde{\rho}_0)$.

So, from Theorem 4.4 and results of work [1] we can conclude that there exists an invertible operator $X \in L(L_p(\Gamma, \rho), L_p^2(\tilde{\Gamma}_0, \tilde{\rho}_0))$ such that $XAX^{-1} \in \Sigma_2(\tilde{\Gamma}_0, \tilde{\rho}_0)$ for any operator $A \in \Sigma(\Gamma, \rho, V)$. Let's designate by $A(t, \mu)$ $(0 \leq t, \mu \leq 1)$ the symbol of the operator $XAX^{-1} \in \Sigma_2(\tilde{\Gamma}_0, \tilde{\rho}_0)$. The matrix-function $A(t, \mu)$ $(0 \leq t, \mu \leq 1)$ of the fourth order can be naturally called the symbol of the operator A. The symbol $A(t, \mu)$ of the operator $A \in \Sigma(\Gamma, \rho; V)$ we shall denote by $\|a_{jk}(t, \mu)\|_{j,k=1}^2$, where $a_{jk}(t, \mu)$ are matrix-functions of the second order. From properties of the symbols of operators from the algebra $\Sigma_2(\tilde{\Gamma}_0, \tilde{\rho}_0)$ (see [4]) and arguments given above, results the following theorem:

Theorem 4.5. *In order that the operator* $A \in \Sigma(\Gamma, \rho; V)$ *be Noetherian in the space* $L_p(\Gamma, \rho)$ *it is necessary and sufficient that the determinant of its* *symbol be separated from zero. If this condition is satisfied, then*

$$\mathtt{Ind}A = -\frac{1}{2\pi}\left\{\arg\det A(t,\mu)\Big/\det a_{22}(t,0)\det a_{22}(t,0)\right\}_{\substack{0\le t\le 1\\ 0\le\mu\le 1}}.$$

5. EXAMPLE

In this section an example of a singular operator with shift (reversing the orientation of the contour) is given, which shows that if the operator coefficients have discontinuity points, Theorems 4.1 and 4.2 , generally speaking, do not occur. For the sake of simplicity we will consider that $\Gamma = \mathbb{R}^+$, $\lambda = 1$ and $(W\varphi)(x) = \frac{1}{x}\varphi\left(\frac{1}{x}\right)$. Let's consider the operator $R = I + \alpha\chi(x)S_+W$ in the space $L_z(\mathbb{R}^+)$, where $\alpha = \text{const.}$, $\chi(x)$ $(x \in \mathbb{R}_+)$ is the characteristic function of the interval $[1, +\infty)$ and S_+ is the operator of singular integration along $\mathbb{R}^+$. The operator R belongs to the algebra $\Sigma(\mathbb{R}^+; W)$. Its symbol looks as

$$R(t,\mu) = \begin{pmatrix} a_{11}(t,\mu) & 0 \\ 0 & a_{22}(t,\mu) \end{pmatrix},$$

where $a_{22}(t,\mu) = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$, $a_{11}(t,\mu) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ if $0 < t < 1$ and

$$a_{22}(t,\mu) = \begin{cases} \begin{pmatrix} 1 & -\alpha(2\mu-1) \\ 0 & 1 \end{pmatrix}, & \text{if } t = 1, \\ \begin{pmatrix} 1 + i\alpha\sin\pi\mu & -\alpha\cos\pi\mu \\ 0 & 1 \end{pmatrix}, & \text{if } t = 0 \end{cases}$$

in the space $L_2(\mathbb{R}^+)$.

Consequently, $\det R(t,\mu) = 1$ if $t \neq 0$ and $\det R(t,\mu) = 1 + \mathrm{i}\alpha\sin\pi\mu$. If consider $\alpha = -\mathrm{i}$, then $\det R(t,\mu) \neq 0$ $(0 \le t,\mu \le 1)$, so, the operator $I - \mathrm{i}\chi(t)S_+W$ is Noetherian in $L_2(\mathbb{R}^+)$. If consider $\alpha = \mathrm{i}$, then $\det R(0, 1/2) = 0$ and, therefore, the operator $I + \mathrm{i}\chi(t)SW$ is not Noetherian in $L_2(\mathbb{R}^+)$.

Let $R_- = I - \mathrm{i}\chi(t)SW$ and $R_+ = I + \mathrm{i}\chi(t)SW$; as R_- is Noetherian, but R_+ is not Noetherian, then according to the equality (4.2), the operator R_W is not Noetherian in $L_2^2(\mathbb{R}^+)$. So, for the operator R Theorem 4.1 is not true. Examples constructing is finished.

The analogous example for the space $L_2(-1,1)$ and $(W\varphi)(t) = \varphi(-t)$ was constructed in [1]. Note also that with the help of the operator M_2, determined by the equality (4.12), the constructed example can be reduced to the corresponding example from [1]. The case, when unlimited contour has corners [11, 12], will be studied in other authors publications.

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