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ACCELERATING THE CONVERGENCE OF THE ITERATIVE METHODS OF INTERPOLATORY TYPE

ION PĂVĂLOIU*

Abstract. In this paper we deal with iterative methods of interpolatory type, for solving nonlinear equations in Banach spaces. We show that the convergence order of the iterations may considerably grow if the nodes are properly controlled.

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1. INTRODUCTION

Let X be a Banach space, $D \subseteq X$ a subset, and $f : D \to X$ a nonlinear mapping. Consider the equation

$$f(x) = \theta, \tag{1}$$

where $\theta \in X$ is the zero vector of X.

Regarding f we make the following assumptions:

a) $f : D \to f(D)$ is a one to one mapping;
b) equation (1) has a solution $x^* \in D$;
c) the operator f is Fréchet differentiable on D and $f'(x) \neq \theta_1$, where θ_1 is the null linear operator.

In order to accelerate the convergence of the iterative methods of interpolatory type, we also consider an equation, equivalent to (1), of the form

$$x = \varphi(x), \tag{2}$$

where $\varphi : D \to D$.

We make the following assumptions regarding φ:

a′) φ is p times differentiable on the whole set D, for some $p \in \mathbb{N}$;
b′) we have $\varphi^{(i)}(\theta) = \theta_i$, $i = \overline{1, p-1}$, but $\varphi^{(p)}(\theta) \neq \theta_p$, where θ_i denotes the i-linear operator. Also, $\|\varphi^{(p)}(x)\| \leq L$, $\forall x \in D$, for some $L \geq 0$.

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†"T. Popoviciu" Institute of Numerical Analysis, P.O. Box 68-1, Cluj-Napoca, Romania, e-mail: pavaloiu@ictp.acad.ro.

Let $x_0, x_1, \dots, x_n \in D$ and $y_0 = f(x_0), y_1 = f(x_1), \dots, y_n = f(x_n)$. For $f^{-1} : f(D) \to D$, the Newton identity holds (see, e.g., [3], [4]):

$$f^{-1}(y) = x_0 + [y_0, y_1; f^{-1}](y - y_0) + [y_0, y_1, y_2; f^{-1}](y_1 - y)(y - y_0) + \dots + [y_0, \dots y_n; f^{-1}](y - y_{n-1}) \dots (y - y_0) + [y, y_0, \dots y_n; f^{-1}](y - y_n) \dots (y - y_0). \tag{3}$$

By hypotheses a) and b) we get the relation:

$$x^* = f^{-1}(\theta). \tag{4}$$

The relation above and (3) for $y = \theta$ attract that

$$x^* = x_0 + \sum_{j=1}^{n} (-1)^j [y_0, \dots y_j; f^{-1}] y_{j-1} \dots y_0 + (-1)^{n+1} [\theta, y_0, \dots y_n; f^{-1}] y_n \dots y_0, \tag{5}$$

whence we deduce an approximation u for x^*, of the form:

$$u = x_0 + \sum_{j=1}^{n} (-1)^j [y_0, \dots y_j; f^{-1}] y_{j-1} \dots y_0. \tag{6}$$

The error for this approximation is bounded by

$$\|x^* - u\| \le \|[\theta, y_0, \dots y_n; f^{-1}]\| \cdot \|y_n\| \dots \|y_0\|, \tag{7}$$

where the norm of $[\theta, y_0, \dots y_n; f^{-1}]$ is considered in the space of $n+1$-linear operators.

In [2] it is shown that the convergence order of the iterative methods given by (6) cannot be greater than 2, even if the number of the interpolation nodes is arbitrarily increased. However, the convergence order can be increased if we use the auxilliary function φ considered above.

Let $x_0 \in D$ be an initial approximation to x^*. Denote $u_0^0 = x_0, u_1^0 = \varphi(u_0^0), \dots, u_n^0 = \varphi(u_{n-1}^0)$ and $y_0^0 = f(u_0^0), y_1^0 = f(u_1^0), \dots, y_n^0 = f(u_n^0)$. Replacing in (6) the interpolation nodes by y_i^0, $i = \overline{0, n}$, we obtain for x^* a first approximation, denoted by x_1:

$$x_1 = x_0 + \sum_{j=1}^{n} (-1)^j [y_0^0, \dots y_j^0; f^{-1}] y_{j-1}^0 \dots y_0^0. \tag{8}$$

If $x_i \in D$ is an approximation for x^*, $i \in \mathbb{N}$, $n \ge 1$, then we obtain the next approximation in the following way. Denote $u_0^i = x_i, u_1^i = \varphi(u_0^i), \dots, u_n^i = \varphi(u_{n-1}^i)$ and $y_0^i = f(u_0^i), \dots, y_n^i = f(u_n^i)$. Analogously to (8) we get

$$x_{i+1} = x_i + \sum_{j=1}^{n} (-1)^j [y_0^i, \dots y_j^i; f^{-1}] y_{j-1}^i \dots y_0^i, \tag{9}$$

with the error

$$\|x^* - x_{i+1}\| \le \|[\theta, y_0^i, \dots y_n^i; f^{-1}]\| \cdot \|y_n^i\| \dots \|y_0^i\|. \tag{10}$$

In the following we shall analyze some particular instance of (9).

If we take $n = 1$ in (9), we get

$$x_{i+1} = x_i - [y_0^i, y_1^i; f^{-1}]y^i, \quad i = 0, 1, \ldots . \tag{11}$$

Taking into account the identities $[y_0^i, y_1^i; f^{-1}] = [x_i, \varphi(x_i); f]^{-1}$ and $y_0^i = f(x_i)$, $y_1^i = f(\varphi(x_i))$, we notice that we are lead to the Steffensen method:

$$x_{i+1} = x_i - [x_i, \varphi(x_i); f]^{-1} f(x_i). \tag{12}$$

If we take $n = 2$ and recall that

(13)

$$\begin{aligned}[y_0^i, y_1^i, y_2^i; f^{-1}]y_0^i y_1^i = & -[x_i, \varphi(\varphi(x_i)); f]^{-1}[x_i, \varphi(x_i), \varphi(\varphi(x_i)); f] \\ & [x_i, \varphi(x_i); f]^{-1} f(x_i)[\varphi(x_i), \varphi(\varphi(x_i)); f]^{-1} f(\varphi(x_i)),\end{aligned}$$

$i = 0, 1, \ldots$, and we get

$$\begin{aligned}x_{i+1} = x_i & - [x_i, \varphi(x_i); f]^{-1} f(x_i) \\ & - [x_i, \varphi(\varphi(x_i)); f]^{-1}[x_i, \varphi(x_i), \varphi(\varphi(x_i)); f] \\ & [x_i, \varphi(x_i); f]^{-1} f(x_i)[\varphi(x_i), \varphi(\varphi(x_i)); f]^{-1} f(\varphi(x_i)),\end{aligned} \tag{14}$$

a corrected Steffensen method.

In [5] we have studied the local convergence of the Steffensen method (12). In the following we shall study the local convergence of the general method (9). We shall show that the convergence order considerably grows even for $p = 1$, under condition a$'$).

2. LOCAL CONVERGENCE

From (10), using the finite growth formula (see, e.g., [4]), we get

$$\|x^* - x_{i+1}\| \le MK^{n+1}\|x^* - u_n^i\| \ldots \|x^* - u_0^i\|, \quad i = 0, 1, \ldots, \tag{15}$$

where we have assumed that there exist $M, K > 0$ such that

$$\|[\theta, v_1, \ldots, v_n; f^{-1}]\| \le M, \quad \forall v_i \in D, i = \overline{0, n}, \tag{16}$$

$$\|f'(x)\| \le K, \quad \forall x \in D. \tag{17}$$

From the hypotheses a$'$) and b$'$), using the Taylor formula we get

$$\|x^* - u_s^i\| \le l^{\delta_s}\|x^* - x_i\|^{p^s}, \quad s = \overline{1, n}, \tag{18}$$

where $l = \frac{L}{p!}$ and $\delta_s = \sum_{j=0}^{s-1} p^j$.

We make the following notations:

$$\alpha = \begin{cases} \sum\limits_{s=1}^{n} \frac{p^s-1}{p-1}, & p > 1 \\ \frac{n(n+1)}{2}, & p = 1 \end{cases} \tag{19}$$

$$\beta = \begin{cases} \frac{p^{n+1}-1}{p-1}, & p > 1 \\ n+1, & p = 1 \end{cases} \tag{20}$$

$$q = MK^{n+1}l^{\alpha}. \tag{21}$$

Using (18), relations (15) become

$$\|x^* - x_{i+1}\| \leq q\|x^* - x_i\|^{\beta}, \quad i = 0, 1, \ldots . \tag{22}$$

Multiplying these inequalities by $q^{\frac{1}{\beta-1}}$ and denoting $\eta_i = q^{\frac{1}{\beta-1}}\|x^* - x_i\|$, $i = 0, 1, \ldots$, we get:

$$\eta_{i+1} \leq \eta_i^{\beta}, \quad i = 0, 1, \ldots . \tag{23}$$

Taking into account the above considerations, we obtain the following result.

Theorem 2.1. *If the hypotheses* a)–c), a′), b′) *hold and, moreover,*

i. $\eta_0 \leq 1$;

ii. $S = \{x \in X : \|x - x^*\| \leq q^{\frac{1}{1-\beta}}\eta_0\} \subseteq D$,

then the elements of the sequence $(x_m)_{m\geq 0}$ *generated by* (9) *remain in the set* D, $\lim x_m = x^*$, *and for all* $m = 1, 2, \ldots,$ *we have*

$$\|x^* - x_m\| \leq q^{\frac{1}{1-\beta}}\eta_0^{\beta^m}. \tag{24}$$

Proof. By (23) and i. it follows

$$\|x^* - x_1\| \leq q^{\frac{1}{1-\beta}}\eta_0^{\beta},$$

which shows that $x_1 \in S$.

Let $x_i \in S$ and $\|x^* - x_i\| \leq q^{\frac{1}{1-\beta}}\eta_0^{\beta^i}$, so

$$\|x^* - x_{i+1}\| \leq q^{\frac{1}{1-\beta}}\eta_0^{\beta^{i+1}} < q^{\frac{1}{1-\beta}}\eta_0,$$

which shows that $x_{i+1} \in S$.

Relations (24) can be easily proved, and further, taking into consideration i., we obviously get $\lim x_m = x^*$. □

Remark 2.1. By (24) it follows that the convergence order of method (9) is at least β.

In case of the particular methods (12) and (14) we get the following results.

Corollary 2.1. *In the case of the Steffensen method, we obtain the well known result* (*see, e.g.,* [5]) *for* $\alpha = 1$ *and* $\beta = 1 + p$ *if* $p > 1$ *and* $\beta = 2$ *if* $p = 1$.

Corollary 2.2. *In the case of method* (14) *we get*

$$\alpha = \begin{cases} p+2, & p>1 \\ 3, & p=1 \end{cases}$$

$$\beta = \begin{cases} 1+p+p^2, & p>1 \\ 3, & p=1. \end{cases}$$

We conclude that the iterative methods of interpolatory type may attain a substantially higher convergence order if the nodes are properly controlled.

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