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ABOUT A GENERAL PROPERTY FOR A CLASS OF LINEAR POSITIVE OPERATORS AND APPLICATIONS

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Abstract. In this paper we demonstrate a general property for a class of linear positive operators. By particularization, we obtain the convergence and the evaluation for the rate of convergence in term of the first modulus of smoothness for the Bernstein operators, Durrmeyer operators, Kantorovich operators and Bleimann, Butzer and Hahn operators.

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1. INTRODUCTION

In this section, we recall some notions and results which we will use in this article (see [10]).

We consider $I \subset \mathbb{R}$, I an interval and we shall use the function sets: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on I, $B(I) = \{f | f : I \to \mathbb{R},\ f \text{ bounded on } I\}$, $C(I) = \{f | f : I \to \mathbb{R},\ f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$. For $x \in I$, consider the function $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$, for any $t \in \mathbb{R}$.

Let a, b, a', b' be real numbers, $I \subset \mathbb{R}$ interval, $a < b$, $a' < b'$, $[a, b] \subset I$, $[a', b'] \subset I$ and $[a, b] \cap [a', b'] \neq \emptyset$. For any non zero natural number m, consider the functions $\varphi_{m,k} : I \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in [a', b']$, for any $k \in \{0, 1, \ldots, m\}$ and the linear positive functionals $A_{m,k} : E([a, b]) \to \mathbb{R}$ for any $k \in \{0, 1, \ldots, m\}$.

For m a non zero natural number, define the operator $L_m^* : E([a, b]) \to F(I)$ by

$$(L_m^* f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f), \tag{1}$$

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for any $f \in E([a,b])$, for any $x \in I$ and for a natural number i, define $T^*_{m,i}$ by

$$(T^*_{m,i}L^*_m)(x) = m^i(L^*_m\psi^i_x)(x) = m^i\sum_{k=0}^{m}\varphi_{m,k}(x)A_{m,k}(\psi^i_x), \tag{2}$$

for any $x \in [a,b]\cap[a',b']$.

In the following, let s be a fixed natural number, s even and we suppose that the operators $\left(L^*_m\right)_{m\geq 1}$ verify the conditions: there exists the smallest $\alpha_s, \alpha_{s+1} \in [0,\infty)$ so that

$$\lim_{m\to\infty}\frac{(T^*_{m,j}L^*_m)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R} \tag{3}$$

for any $x \in [a,b]\cap[a',b']$, $j \in \{s, s+2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{4}$$

If $I \subset \mathbb{R}$ is a given interval and $f \in C_B(I)$, then the first order modulus of smoothness of f is the function $\omega_1 : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega_1(f;\delta) = \sup\left\{|f(x') - f(x')| : x', x'' \in I, |x' - x''| \leq \delta\right\}.$$

In [10] are the results contained in the following.

Proposition 1. *For m a non zero natural number, the L^*_m operators are linear and positive.*

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a function. If f is a s times derivable on $[a,b]$, the function $f^{(s)}$ is continuous on $[a,b]$ and there exists a non zero natural number $m(s)$ and a real numbers k_s, k_{s+2} so that for any natural number m, $m \geq m(s)$ and for any $x \in [a,b]\cap[a',b']$, we have*

$$\frac{(T^*_{m,j}L^*_m)(x)}{m^{\alpha_j}} \leq k_j, \tag{5}$$

where $j \in \{s, s+2\}$, then

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[(L^*_m f)(x) - \sum_{i=0}^{s}\frac{f^{(i)}(x)}{m^i i!}\,(T^*_{m,i}L^*_m)(x)\right] = 0 \tag{6}$$

for any $x \in [a,b]\cap[a',b']$, the convergence given in (6) is uniform on $[a,b]\cap[a',b']$ and

$$m^{s-\alpha_s}\left|(L^*_m f)(x) - \sum_{i=0}^{s}\frac{f^{(i)}(x)}{m^i i!}\,(T^*_{m,i}L^*_m)(x)\right| \leq \tag{7}$$

$$\leq \frac{1}{s!}\,(k_s + k_{s+2})\omega_1\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right),$$

for any $x \in [a,b]\cap[a',b']$, for any natural number m, $m \geq m(s)$.

2. PRELIMINARIES

In this section, we recall some operator definitions which we will use in this article. In the following, let m be a non zero natural number.

Let $B_m : C([0,1]) \to C([0,1])$ the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\tfrac{k}{m}\right), \tag{8}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein defined as follows

$$p_{m,k}(x) = \tbinom{m}{k} x^k (1-x)^{m-k} \tag{9}$$

for any $x \in [0,1]$ and for any $k \in \{0, 1, \ldots, m\}$.

The operators $M_n : L_1([0,1]) \to C([0,1])$ are defined for any function $f \in L_1([0,1])$ by

$$(M_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) \mathrm{d}t, \tag{10}$$

for any $x \in [0,1]$, are named Durrmeyer operators, introduced in 1967 by J. L. Durrmeyer in [8] and were studied in 1981 by M. M. Derriennic in [6].

The operators $K_m : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t) \mathrm{d}t, \tag{11}$$

for any $x \in [0,1]$, are named Kantorovich operators, introduced and studied in 1930 by L. V. Kantorovich (see [13]).

In 1980, G. Bleimann, P. L. Butzer and L. Hahn introduced in [5] a sequence of linear positive operators $(L_m)_{m \geq 1}$, $L_m : C_B([0,\infty)) \to C_B([0,\infty))$, defined for any function $f \in C_B([0,\infty))$ by

$$(L_m f)(x) = \tfrac{1}{(1+x)^m} \sum_{k=0}^{m} \tbinom{m}{k} x^k f\left(\tfrac{k}{m+1-k}\right), \tag{12}$$

for any $x \in [0,\infty)$, for any non zero natural number m. These operators are named Bleimann, Butzer and Hahn operators.

3. MAIN RESULTS

In the following, in Theorem 2 we consider $s = 0$. For the operators recall in the preliminaries we have $\alpha_0 = 0$ and $\alpha_2 = 1$. Then Theorem 2 becomes:

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a function. If f is continuous on $[a,b]$ and there exists a non zero natural number $m(0)$ and a real numbers k_0, k_2 so* *that for any natural number m, $m \geq m(0)$ and for any $x \in [a,b] \cap [a',b']$, we have*

$$(T^*_{m,0} L^*_m)(x) \leq k_0 \tag{13}$$

and

$$\frac{(T^*_{m,2} L^*_m)(x)}{m} \leq k_2, \tag{14}$$

then

$$\lim_{m\to\infty} \Big[(L^*_m f)(x) - f(x)(T^*_{m,0} L^*_m)(x) \Big] = 0 \tag{15}$$

for any $x \in [a,b] \cap [a',b']$, the convergence given in (15) is uniform on $[a,b] \cap [a',b']$ and

$$\Big| (L^*_m f)(x) - f(x)(T^*_{m,0} L^*_m)(x) \Big| \leq (k_0 + k_2)\omega_1(f; \tfrac{1}{\sqrt{m}}), \tag{16}$$

for any $x \in [a,b] \cap [a',b']$, for any natural number m, $m \geq m(0)$.

Proof. Theorem 3 is a consequence of Theorem 2. □

In Applications 4, 6, 8 and 10 we consider $a = a' = 0$, $b = b' = 1$ and $\varphi_{m,k} = p_{m,k}$, for any m, k natural numbers, $m \neq 0$ and $k \in \{0, 1, \ldots, m\}$.

Application 4. *For any non zero natural number m, we consider the functionals $A_{m,k} : C([0,1]) \to \mathbb{R}$, $A_{m,k}(f) = f(\frac{k}{m})$, for any $k \in \{0, 1, \ldots, m\}$ and for any $f \in C([0,1])$.*

In this application, we obtain the Bernstein operators and $(T^*_{m,0} B_m)(x) = T_{m,0}(x) = 1$, $(T^*_{m,2} B_m)(x) = T_{m,2}(x) = mx(1-x) \leq \frac{1}{4}m$, for any $x \in [0,1]$, for any non zero natural number m, so $k_0 = 1$, $k_2 = \frac{1}{4}$ (see [9] or [13]).

Because the condition (13) and (14) take place, Theorem 3 is announced thus:

Theorem 5. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function on $[0,1]$. Then*

$$\lim_{m\to\infty} (B_m f)(x) = f(x) \tag{17}$$

for any $x \in [0,1]$, the convergence given in (17) is uniform on $[0,1]$ and

$$\big| (B_m f)(x) - f(x) \big| \leq \tfrac{5}{4}\, \omega_1(f; \tfrac{1}{\sqrt{m}}) \tag{18}$$

for any $x \in [0,1]$, for any non zero natural number m.

Application 6. *Let m be a non zero natural number, the functionals $A_{m,k} : L_1([0,1]) \to \mathbb{R}$,*

$$A_{m,k}(f) = (m+1) \int_0^1 p_{m,k}(t) f(t) \mathrm{d}t,$$

for any $f \in L_1([0,1])$, for any $k \in \{0, 1, \ldots, m\}$.

In this case, we obtain the Durrmeyer operators. We have $(T^*_{m,0} M_m)(x) = 1$, $(T^*_{m,2} M_m)(x) = m^2 \frac{2(m-3)x(1-x)+2}{(m+2)(m+3)}$, for any $x \in [0,1]$, for any non zero *natural number m, so $k_0 = 1$, $k_2 = \frac{1}{4}$ for any $m \in \mathbb{N}$, $m \geq 3$ (see [10]). Then, we have:*

THEOREM 7. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function on $[0,1]$. Then*

$$\lim_{m\to\infty} (M_m f)(x) = f(x) \tag{19}$$

for any $x \in [0,1]$, the convergence given in (19) is uniform on $[0,1]$ and

$$\big|(M_m f)(x) - f(x)\big| \leq \tfrac{5}{4}\,\omega_1\big(f; \tfrac{1}{\sqrt{m}}\big), \tag{20}$$

for any $x \in [0,1]$, for any natural number m, $m \geq 3$.

APPLICATION 8. *For any non zero natural number m, consider the functionals $A_{m,k} : L_1([0,1]) \to \mathbb{R}$,*

$$A_{m,k}(f) = (m+1)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)\mathrm{d}t$$

for any $f \in L_1([0,1])$ and for any $k \in \{0,1,\ldots,m\}$. In this case, we obtain the Kantorovich operators. We have

$$(T^*_{m,0}K_m)(x) = 1, (T^*_{m,2}K_m)(x) = \big(\tfrac{m}{m+1}\big)^2 \tfrac{(1-x)^3+x^3+3mx(1-x)}{3},$$

for any $x \in [0,1]$, for any non zero natural number m and $k_0 = 1$, $k_2 = 1$ (see [10]).

THEOREM 9. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function on $[0,1]$. Then*

$$\lim_{m\to\infty} (K_m f)(x) = f(x) \tag{21}$$

for any $x \in [0,1]$, the convergence given in (21) is uniform on $[0,1]$ and

$$\big|(K_m f)(x) - f(x)\big| \leq 2\omega_1\big(f; \tfrac{1}{\sqrt{m}}\big), \tag{22}$$

for any $x \in [0,1]$, for any non zero natural number m.

APPLICATION 10. *In this case, for any non zero natural number m, we consider $\varphi_{m,k}(x) = \binom{m}{k}\frac{x^k}{(1+x)^m}$, for any $x \in [0,\infty)$ and $A_{m,k}(f) = f\big(\frac{k}{m+1-k}\big)$, for any $k \in \{0,1,\ldots,m\}$. Then, the operators defined by (1) become the Bleimann, Butzer and Hahn operators. If $b > 0$, we have $k_0 = 1$, $k_2 = 4b(1+b)^2$, for any $x \in [0,b]$, for any natural number m, $m \geq 24(1+b)$ (see [12]).*

THEOREM 11. *If $f \in C([0,\infty))$, then*

$$\lim_{m\to\infty} (L_m f)(x) = f(x) \tag{23}$$

for any $x \in [0,\infty)$, the convergence being uniform on each compact subinterval $[0,b]$ of $[0,\infty)$ and

$$\big|(L_m f)(x) - f(x)\big| \leq \big[1 + 4b(1+b)^2\big]\omega_1\big(f; \tfrac{1}{\sqrt{m}}\big) \tag{24}$$

for any $x \in [0,b]$, for any natural number m, $m \geq 24(1+b)$.

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