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THE MANN-ISHIKAWA ITERATIONS AND THE MANN-ISHIKAWA ITERATIONS WITH ERRORS ARE EQUIVALENT MODELS DEALING WITH A NON-LIPSCHITZIAN MAP

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Abstract. The Mann-Ishikawa iterations and the Mann-Ishikawa iterations with errors are equivalent models for several classes of non-Lipschitzian operators.

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1. INTRODUCTION

The first theorem on fixed point theory was that of Banach [1]. Banach's theorem has been applied in many different areas. For example, it can be used to prove the existence and uniqueness of solutions of certain differential equations. It can also be used to provide a proof of the implicit function theorem. One drawback of the Banach theorem is that the hypothesis requires the map to be continuous at each point of the space. In 1968 Kannan [5] introduced a contractive inequality for a map that need not be continuous at every point, but which, like Banach's theorem, had a unique fixed point, which could be obtained by repeated function iteration. Following Kannan's paper there appeared a multitude of papers involving contractive inequalities that did not require continuity of the map at every point. For nonexpansive maps, however, function iteration need not converge to a fixed point. Iteration procedures were then introduced to obtain fixed points for some nonexpansive maps for which functions iteration fails to converge to a fixed point. The two most famous are those of Mann [7] and Ishikawa [4].

Let X be a space, T a selfmap of X. Mann iteration is defined by $x_0 \in X$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \tag{1}$$

where $\alpha_0 = 1,\ 0 \leq \alpha_n \leq 1,\ n \geq 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty$, and $\lim_{n\to\infty} \alpha_n = 0$.

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The Ishikawa iteration process is defined by $x_0 = y_0 \in X$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \qquad (2)$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0,$$

where $0 \leq \alpha_n, \beta_n \leq 1$,

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0, \text{ and} \sum_{n=0}^{\infty} \alpha_n = \infty. \qquad (3)$$

In his original paper, Ishikawa [4] placed the restriction that $\alpha_n \leq \beta_n$. The first author [10] observed that the more general condition

$$0 \leq \alpha_n, \beta_n \leq 1 \qquad (4)$$

could be used for many classes of maps. Since then many papers have been written which extend theorems on Mann iteration to those of Ishikawa iteration, using condition (3).

In a series of papers, including [11], [12], [13], [15], [14] and [16], the authors have shown that Mann and Ishikawa iterations, with the same $\{\alpha_n\}$, and using (3), are equivalent for various independent classes of functions.

Recently definitions were given for Ishikawa iteration with errors. (See, e.g., [6] and [18].) For bounded maps these two processes are equivalent. (See Remarks 1 and 2 below.)

In spite of criticisms in Math. Reviews of papers dealing with iterations with errors, made by both C. E. Chidume (MR: 1999g:47144, 2004d:47111) and the first author (MR 2003a: 47129), papers continue to be published involving Mann or Ishikawa iterations with errors. In this paper we show that, for certain maps with bounded range, that Mann and Ishikawa iterations are equivalent to the corresponding iteration processes with errors.

The *Mann iteration with errors* introduced in [6] is given by the following relation

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n + e_n. \qquad (5)$$

The error $\{e_n\} \subset X$ satisfies

$$\sum_{n=0}^{\infty} \|e_n\| < \infty. \qquad (6)$$

In [6], *Ishikawa iteration with errors* is defined as

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T v_n + p_n, \qquad (7)$$
$$v_n = (1-\beta_n)u_n + \beta_n T u_n + q_n.$$

The errors $\{p_n\}, \{q_n\} \subset X$ satisfy

$$\sum_{n=0}^{\infty} \|p_n\| < \infty, \ \sum_{n=0}^{\infty} \|q_n\| < \infty. \qquad (8)$$

If $e_n = 0$, respectively $p_n = q_n = 0, \forall n \in \mathbb{N}$, then we deal with Mann and Ishikawa iteration.

Another, Ishikawa iteration with errors is introduced in [18]:

$$(9) \qquad \begin{aligned} u_{n+1} &= a_n u_n + b_n T v_n + c_n P_n, \\ v_n &= a'_n u_n + b'_n T u_n + c'_n Q_n, \end{aligned}$$

where $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1,\ a_n, b_n, c_n, a'_n, b'_n, c'_n \in [0,1], \forall n \in \mathbb{N}$, $P_n, Q_n \subset X$, are the errors supposed bounded. Take in (9), $a'_n = 1,\ b'_n = c'_n = 0, \forall n \in \mathbb{N}$, to obtain Mann iteration with errors, see [18]:

$$(10) \qquad u_{n+1} = a_n u_n + b_n T v_n + c_n P_n.$$

The following Remark is from [3].

Remark 1. [3] If T has a bounded range, iteration (7) is equivalent to (9). □

Remark 1 has a simple proof. Take in (7)

$$\beta_n := b'_n + c'_n,\ \alpha_n := b_n + c_n,\ \forall n \in \mathbb{N},$$

to obtain

$$a_n = 1 - (b_n + c_n) = 1 - \alpha_n,\ a'_n = 1 - (b'_n + c'_n) = 1 - \beta_n.$$

Iteration (7) becomes

$$(11) \qquad \begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n T v_n + c_n (P_n - T v_n), \\ v_n &= (1-\beta_n)u_n + \beta_n T u_n + c'_n (Q_n - T u_n), \quad n \in \mathbb{N}. \end{aligned}$$

Denote

$$(12) \qquad \begin{aligned} p_n &:= c_n (P_n - T v_n), \\ q_n &:= c'_n (Q_n - T u_n). \end{aligned}$$

If $\sum_{n=1}^{\infty} c_n < \infty,\ \sum_{n=1}^{\infty} c'_n < \infty$, then using the boundedness conditions for $\{P_n\}, \{Q_n\}$, and the boundedness of $\{Tv_n\}, \{Tu_n\}$, we get (8). Relations (11) and (12) lead to (9).

Remark 2. A similar proof leads to the conclusion that Mann iteration with errors from [18] is equivalent to (5) introduced in [6]. □

The map $J : X \to 2^{X^*}$ given by

$$Jx := \{f \in X^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\|\}, \quad \forall x \in X,$$

is called *the normalized duality mapping.* The Hahn-Banach theorem assures that $Jx \neq \emptyset, \forall x \in X$.

Proposition 1. [16] *If $x \in X$ and $f \in J(x)$, then*

$$(13) \qquad \langle y, f \rangle \leq \|x\|\ \|y\|, \quad \forall y \in X.$$

If X is a real Hilbert space, then $J = Identity$, and inequality (13) is known as the Cauchy-Buniakowski-Schwarz inequality.

Definition 2. *Let X be a real Banach space, map $T : X \to X$ is called strongly pseudocontractive if there exists a $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that*

$$\langle Tx - Ty, j(x-y)\rangle \leq k \|x-y\|^2, \quad \forall x, y \in X. \tag{14}$$

We recall the following results.

Lemma 3. [8] *If X is a real Banach space, then the following relation is true*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in X, \forall j(x+y) \in J(x+y). \tag{15}$$

Lemma 4. [6] *Let $(a_n)_n$ be a nonnegative sequence which satisfies the following inequality*

$$a_{n+1} \leq (1-\lambda_n)a_n + \sigma_n + w_n,$$

where $\lambda_n \in (0,1)$, $w_n \geq 0, \forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\sum_{n=1}^{\infty} w_n < \infty$ and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

Proposition 5. *Let X be a Banach space, $T : X \to X$ a map with $T(X)$ bounded. Then the $\{u_n\}$ and $\{v_n\}$ are bounded. If we take $\beta_n = 0, \forall n \in \mathbb{N}$ in (7), then $\{u_n\}$ given by (5) is bounded.*

Proof. Let $M_1 = \sup\{\|Tx\| : x \in X\}$. Then M_1 is finite because T has bounded range. Set

$$M = \max(M_1, \|u_1\|).$$

We have

$$\begin{aligned}\|u_2\| &\leq (1-\alpha_1)\|u_1\| + \alpha_1 \|Tv_1\| + \|p_1\| \\ &\leq (1-\alpha_1)\|u_1\| + \alpha_1 M_1 + \|p_1\| \\ &\leq (1-\alpha_1) M + \alpha_1 M + \|p_1\| \\ &= M + \|p_1\|.\end{aligned}$$

Supposing that

$$\|u_n\| \leq M + \sum_{k=1}^{n-1} \|p_k\|, \tag{16}$$

we prove that (16) holds for $(n+1)$:

$$\begin{aligned}\|u_{n+1}\| &\leq (1-\alpha_n)\|u_n\| + \alpha_n \|Tv_n\| + \|p_n\| \\ &\leq (1-\alpha_n)\left(M + \sum_{k=1}^{n-1}\|p_k\|\right) + \alpha_n M_1 + \|p_n\| \\ &\leq (1-\alpha_n)\left(M + \sum_{k=1}^{n-1}\|p_k\|\right) +\end{aligned} \tag{17}$$

$$+ \alpha_n \left(M + \sum_{k=1}^{n-1} \|p_k\| \right) + \|p_n\|$$

$$= M + \sum_{k=1}^{n-1} \|p_k\| + \|p_n\| = M + \sum_{k=1}^{n} \|p_k\| .$$

Since $\sum_{k=1}^{\infty} \|p_k\| < \infty$, $\{\|u_n\|\}$ is bounded.

Suppose the following inequality is satisfied for n:

$$\|v_n\| \leq M + \sum_{k=1}^{n} \|p_k\| + \sum_{k=1}^{n} \|q_k\| . \tag{18}$$

We shall prove that (18) holds for $(n+1)$

$$\begin{aligned}
\|v_{n+1}\| &\leq (1-\beta_n) \|u_n\| + \beta_n \|Tu_n\| + \|q_n\| \\
&\leq (1-\beta_n) \left(M + \sum_{k=1}^{n-1} \|p_k\| + \sum_{k=1}^{n-1} \|q_k\| \right) + \beta_n \|Tu_n\| + \|q_n\| \\
&\leq (1-\beta_n) \left(M + \sum_{k=1}^{n-1} \|p_k\| + \sum_{k=1}^{n-1} \|q_k\| \right) + \beta_n M + \|q_n\| \\
&\leq (1-\beta_n) \left(M + \sum_{k=1}^{n-1} \|p_k\| + \sum_{k=1}^{n-1} \|q_k\| \right) \\
&\quad + \beta_n \left(M + \sum_{k=1}^{n-1} \|p_k\| + \sum_{k=1}^{n-1} \|q_k\| \right) + \|q_n\| \\
&= M + \sum_{k=1}^{n} \|p_k\| + \sum_{k=1}^{n} \|q_k\| .
\end{aligned}$$

Since $\sum_{k=1}^{\infty} \|p_k\| < \infty$ and $\sum_{k=1}^{\infty} \|q_k\| < \infty$, $\{\|v_n\|\}$ is bounded. □

2. MAIN RESULT

Denote $F(T) = \{x^* : Tx^* = x^*\}$.

THEOREM 6. *Let X be a real Banach space with a uniformly convex dual space, $T : X \to X$ a strongly pseudocontractive operator, with $T(X)$ bounded. Let $\{\alpha_n\} \in (0,1)$, $\{\beta_n\} \in [0,1)$ satisfy* (3), *and the errors $\{p_n\}, \{q_n\}$ satisfy* (8) *. If $u_0 = x_0 \in X$, then the following are equivalent:*

(i) *Ishikawa iteration* (2) *converges to $x^* \in F(T)$,*
(ii) *Ishikawa iteration with errors* (7) *converges to $x^* \in F(T)$.*

Proof. The strongly pseudocontractive condition implies the uniqueness of x^*. Suppose that x^* and y^* are two distinct fixed points of T. Then

$$\begin{aligned}\langle Tx^* - Ty^*, j(x^* - y^*)\rangle &\leq k\,\|x^* - y^*\|^2, \\ \langle x^* - y^*, j(x^* - y^*)\rangle &\leq k\,\|x^* - y^*\|^2, \\ \|x^* - y^*\|^2 &\leq k\,\|x^* - y^*\|^2,\end{aligned}$$

which implies $\|x^* - y^*\| = 0$, thus $x^* = y^*$.

The sequence $\{x_n\}$ is convergent, thus bounded; also T is bounded. Taking

$$M_1 := \sup\{\{\|x_n\| : n \in \mathbb{N}\}, \{\|Tx\| : x \in X\}\} < \infty,$$

we have

$$\|y_n\| \leq (1-\beta_n)\,\|x_n\| + \beta_n\,\|Tx_n\| \leq (1-\beta_n)M_1 + \beta_n M_1 = M_1.$$

Proposition 5 assures that $\{\|u_n\|\}, \{\|v_n\|\}$ are bounded. Define

$$M := \sup\{M_1, \{\|u_n\|, \|v_n\| : n \in \mathbb{N}\}\} < \infty.$$

Using (15) with

(19)
$$\begin{aligned} x &:= (1-\alpha_n)\,(u_n - x_n), \\ y &:= \alpha_n\,(Tv_n - Ty_n) + p_n, \\ x + y &= u_{n+1} - x_{n+1}, \end{aligned}$$

(13), Definition 2, (14), (2) and (7) we get

$$\|u_{n+1} - x_{n+1}\|^2 =$$

(20)
$$\begin{aligned}
&= \left\|(1-\alpha_n)\,(u_n - x_n) + \alpha_n\,(Tv_n - Ty_n) + p_n\right\|^2 \\
&\leq (1-\alpha_n)^2\,\|u_n - x_n\|^2 + 2\big\langle \alpha_n\,(Tv_n - Ty_n) + p_n, J(u_{n+1} - x_{n+1})\big\rangle \\
&= (1-\alpha_n)^2\,\|u_n - x_n\|^2 + 2\alpha_n\big\langle Tv_n - Ty_n,\ J(u_{n+1} - x_{n+1}) - J(v_n - y_n)\big\rangle \\
&\quad + 2\alpha_n\big\langle Tv_n - Ty_n, J(v_n - y_n)\big\rangle + 2\big\langle p_n,\ J(u_{n+1} - x_{n+1})\big\rangle \\
&\leq (1-\alpha_n)^2\,\|u_n - x_n\|^2 + 2\alpha_n k\,\|v_n - y_n\|^2 \\
&\quad + 2\alpha_n\big\langle Tv_n - Ty_n,\ J(u_{n+1} - x_{n+1}) - J(v_n - y_n)\big\rangle \\
&\quad + 2\big\langle p_n,\ J(u_{n+1} - x_{n+1})\big\rangle \\
&\leq (1-\alpha_n)^2\ \|u_n - x_n\|^2\ +\ 2\alpha_n k\ \|v_n - y_n\|^2 + \\
&\quad +\ 2\ \alpha_n\ \|Tv_n - Ty_n\|\ \|J(u_{n+1} - x_{n+1})\ -\ J(v_n - y_n)\| \\
&\quad + 2\,\|p_n\|\,\|u_{n+1} - x_{n+1}\| \\
&\leq (1-\alpha_n)^2\,\|u_n - x_n\|^2 + 2\alpha_n k\,\|v_n - y_n\|^2 \\
&\quad + 2\alpha_n 2M\,\|J(u_{n+1} - x_{n+1}) - J(v_n - y_n)\| + 2\,\|p_n\|\,2M
\end{aligned}$$

$$\leq (1-\alpha_n)^2 \|u_n - x_n\|^2 + 2\alpha_n k \|v_n - y_n\|^2$$
$$+ 4M\alpha_n \|J(u_{n+1} - x_{n+1}) - J(v_n - y_n)\| + 4M \|p_n\| .$$

Observe that $\{\|Tv_n - Ty_n\|\}$ is bounded. If X has a uniformly convex dual space, then $J(\cdot)$ is uniformly continuous on each bounded set. We prove that

$$J(u_{n+1} - x_{n+1}) - J(v_n - y_n) \to 0, (n \to \infty). \tag{21}$$

To prove (21) it is sufficient to see that

$$\begin{aligned}
&\|(u_{n+1} - x_{n+1}) - (v_n - y_n)\| = \\
&= \|(y_n - x_{n+1}) + (u_{n+1} - v_n)\| \\
&= || \,(1-\beta_n)\, x_n + \beta_n \,Tx_n - \,(1-\alpha_n)\, x_n - \alpha_n Ty_n + \\
&\quad + (1-\alpha_n)\, u_n \;+\; \alpha_n Tv_n + \; p_n - \;(1-\beta_n)\, u_n \;-\; \beta_n Tu_n \;- q_n|| \\
&= ||\alpha_n x_n - \beta_n x_n + \beta_n Tx_n - \alpha_n Ty_n - \alpha_n u_n + \alpha_n Tv_n \\
&\quad + p_n + \beta_n u_n - \beta_n Tu_n - q_n|| \\
&\leq \alpha_n \|x_n\| + \beta_n \|x_n\| + \beta_n \|Tx_n\| + \alpha_n \|Ty_n\| + \alpha_n \|u_n\| \\
&\quad + \alpha_n \|Tv_n\| + \|p_n\| + \beta_n \|u_n\| + \beta_n \|Tu_n\| + \|q_n\| \\
&\leq 4M\alpha_n + 4M\beta_n + \;\|p_n\| \;+\; \|q_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

We define

$$\delta_n := 2M \|J(u_{n+1} - x_{n+1}) - J(v_n - y_n)\| . \tag{22}$$

Again, using (2), (7) and (15) with

$$\begin{aligned}
x &:= (1-\beta_n)\,(u_n - x_n), \\
y &:= \beta_n(Tu_n - Tx_n) + q_n, \\
x + y &= v_n - y_n,
\end{aligned}$$

we get

(23)

$$
\begin{aligned}
&\|v_n - y_n\|^2 = \\
&= \|(1-\beta_n)\,(u_n - x_n) \;+\; \beta_n\,(Tu_n - Tx_n) + q_n\|^2 \\
&\leq (1-\beta_n)^2\;\|u_n - x_n\|^2 \;+\; 2\;\langle \beta_n\;(Tu_n - Tx_n)\;+\;q_n, J(v_n - y_n)\rangle \\
&= (1-\beta_n)^2\;\|u_n - x_n\|^2 \;+\; 2\,\beta_n\;\langle Tu_n - Tx_n\;, J(v_n - y_n)\rangle + \\
&\quad + 2\;\langle\; q_n,\; J(v_n - y_n)\rangle \\
&= (1-\beta_n)^2\;\|u_n - x_n\|^2 \;+\; 2\,\beta_n\;\langle Tu_n - Tx_n\;, J(u_n - x_n)\rangle + \\
&\quad +\; 2\,\beta_n\;\langle Tu_n - Tx_n\;, J(v_n - y_n) - J(u_n - x_n)\rangle + 2\;\langle\; q_n,\; J(v_n - y_n)\rangle \\
&\leq (1-\beta_n)^2\;\|u_n - x_n\|^2 \;+\; 2\,\beta_n\;\|u_n - x_n\|^2\;+ \\
&\quad + 2\,\beta_n\;\|Tu_n - Tx_n\|\;\|J(v_n - y_n) - J(u_n - x_n)\| + 2\;\|q_n\|\;\|v_n - y_n\| \\
&\leq \left(1 + \beta_n^2\right)\;\|u_n - x_n\|^2 \;+\; 4\,\beta_n\,M\;\|J(v_n - y_n) - J(u_n - x_n)\| + 4\;\|q_n\|\;M
\end{aligned}
$$

$$\leq \|x_n - u_n\|^2 + \beta_n \|x_n - u_n\|^2 + 2\beta_n\delta_n + P\|q_n\|$$
$$\leq \|x_n - u_n\|^2 + S\beta_n + 2\beta_n\delta_n + P\|q_n\|,$$

where

$$S := 4M^2,$$
$$P := 4M.$$

Replacing (22) and (23) in (20), we obtain

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2 \leq \\
&\leq (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n k \left(\|x_n - u_n\|^2 + S\beta_n + 2\beta_n\delta_n + P\|q_n\|\right) \\
&\quad + 2\alpha_n\delta_n + 4M\|p_n\| \\
&= \left(1 - 2(1-k)\alpha_n + \alpha_n^2\right)\|x_n - u_n\|^2 + \\
&\quad + \alpha_n\left(2k\left(S\beta_n + 2\beta_n\delta_n + P\|q_n\|\right) + 2\delta_n\right) + 4M\|p_n\|.
\end{aligned}$$

Condition $k \in (0,1)$ implies $(1-k) \in (0,1)$. From $\lim_{n\to\infty} \alpha_n = 0$ we get

$$\exists n_0 \in \mathbb{N}, \forall n \geq n_0 : \alpha_n \leq (1-k).$$

Thus

$$1 - 2(1-k)\alpha_n + \alpha_n^2 \leq 1 - 2(1-k)\alpha_n + (1-k)\alpha_n = 1 - (1-k)\alpha_n.$$

Hence

$$\|x_{n+1} - u_{n+1}\|^2 \leq (1 - (1-k)\alpha_n)\|x_n - u_n\|^2 + \sigma_n + w_n,$$

where

$$\sigma_n := \alpha_n \left(2k\left(S\beta_n + 2\beta_n\delta_n + P\|q_n\|\right) + 2\delta_n\right),$$
$$w_n := P\|p_n\|.$$

Using Lemma 4, with

$$a_n := \|x_n - u_n\|^2, \quad \lambda_n := (1-k)\alpha_n \in (0,1), \tag{24}$$

we obtain $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \|x_n - u_n\|^2 = 0$; i.e.

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{25}$$

Let x^* be the fixed point of T. Suppose that $\lim_{n\to\infty} x_n = x^*$. The inequality

$$0 \leq \|x^* - u_n\| \leq \|x_n - x^*\| + \|x_n - u_n\|$$

and (25) imply that $\lim_{n\to\infty} u_n = x^*$. Similarly, $\lim_{n\to\infty} u_n = x^* \Rightarrow \lim_{n\to\infty} x_n = x^*$. □

The Mann iteration and the Mann iteration with errors are equivalent.

Theorem 7. *Let X be a Banach space with a uniformly convex dual, $T: X \to X$ a strongly pseudocontractive map with $T(X)$. Let $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and suppose that the errors satisfy (6). If $u_0 = x_0 \in X$, then the following are equivalent:*

(i) *Mann iteration (1) converges to $x^* \in F(T)$,*
(ii) *Mann iteration with errors (5) converges to the same $x^* \in F(T)$.*

Proof. Set $\beta_n = 0, \forall n \in \mathbb{N}$, in (2) and $\beta_n = 0, q_n = 0, \forall n \in \mathbb{N}$, in (7). The conclusion follows from Theorem 6. □

We already know from [11] that (1) and (2) are equivalent.

Theorem 8. [11] *Let X be a real Banach space with a uniformly convex dual, $T: X \to X$ a strongly pseudocontractive operator with $T(X)$ bounded. Let $\{\alpha_n\} \in (0,1)$, $\{\beta_n\} \in [0,1)$ satisfy (3). Then for $u_0 = x_0 \in B$ the following are equivalent:*

(i) *Mann iteration (1) converges to the fixed point of T;*
(ii) *Ishikawa iteration (2) converges to the fixed point of T.*

The above results lead to the following conclusion.

Corollary 9. *Let X be a real Banach space with a uniformly convex dual and $T: X \to X$ a strongly pseudocontractive operator, with $T(X)$ bounded. Let $\{\alpha_n\} \in (0,1)$, $\{\beta_n\} \in [0,1)$ to satisfy (3). Errors $\{p_n\}, \{q_n\}$ satisfy (8). If $u_0 = x_0 \in X$, then the iterations given by (1), (2), (7), (5), (10), and (9) are equivalent.*

3. THE ACCRETIVE AND STRONGLY ACCRETIVE CASE

Definition 10. *Let $S: X \to X$. The map S is called strongly accretive if there exists $j(x-y) \in J(x-y)$ and $k \in (0,1)$ such that*

$$\langle Sx - Sy, j(x-y)\rangle \geq k \|x-y\|^2 . \tag{26}$$

for all $x, y \in X$.

The map S is called accretive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y)\rangle \geq 0. \tag{27}$$

for all $x, y \in X$.

Given the equation $Sx = f$, where S is a strongly accretive map and f is a given point in X, consider iterations (2) and (7) with $Tx := f + (I-S)x$:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n (f + (I-S)y_n), \\ y_n &= (1-\beta_n)x_n + \beta_n (f + (I-S)x_n), \end{aligned} \tag{28}$$

and

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n\left(f + (I-S)v_n\right) + p_n, \\ v_n &= (1-\beta_n)u_n + \beta_n\left(f + (I-S)u_n\right) + q_n. \end{aligned} \tag{29}$$

THEOREM 11. *Let X be a Banach space with a uniformly convex dual, $S : X \to X$ a continuous and strongly accretive map, let $(I-S)$ with bounded range. Let $\{\alpha_n\} \in (0,1)$, $\{\beta_n\} \in [0,1)$ satisfy* (3) *and let errors $\{p_n\}, \{q_n\}$ satisfy* (8). *If $u_0 = x_0 \in B$, then the following are equivalent:*

(i) *Ishikawa iteration* (28) *converges to $x^* \in F(T)$, which is the solution of $Sx = f$,*

(ii) *Ishikawa iteration with errors* (29) *converges to $x^* \in F(T)$, which is the solution of $Sx = f$.*

Proof. In Theorem 6 set

$$Tx := f + (I-S)\,x, \quad \forall x \in X. \tag{30}$$

A fixed point for T will be a solution for $Sx = f$. Note that T given by (30) has bounded range. The map S is a strongly accretive map, since for all $j(x-y) \in J(x-y)$ we have

$$\begin{aligned} \langle Sx - Sy, j(x-y)\rangle &\geq k\,\|x-y\|^2 \Leftrightarrow \\ \langle -Sx - (-Sy), j(x-y)\rangle &\leq -k\,\|x-y\|^2 \Leftrightarrow \\ \|x-y\|^2 + \langle -Sx - (-Sy), j(x-y)\rangle &\leq -k\,\|x-y\|^2 + \|x-y\|^2 \Leftrightarrow \\ \langle x-y, j(x-y)\rangle + \langle -Sx - (-Sy), j(x-y)\rangle &\leq (1-k)\,\|x-y\|^2 \Leftrightarrow \\ \langle f + x - Sx - (f + y - Sy), j(x-y)\rangle &\leq (1-k)\,\|x-y\|^2 \Leftrightarrow \\ \langle Tx - Ty, j(x-y)\rangle &\leq (1-k)\,\|x-y\|^2. \end{aligned}$$

Setting $k' := (1-k) \in (0,1)$ in (14), we get the definition of a strongly pseudocontractive map. □

Set $\beta_n = 0, \forall n \in \mathbb{N}$ in (28) and $\beta_n = 0, q_n = 0, \forall n \in \mathbb{N}$ in (29) to obtain the Mann iteration and the Mann iteration with errors. Consider iterations (1) and (5) with $f + (I-S)x$ instead of Tx :

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\left(f + (I-S)y_n\right), \tag{31}$$

and

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\left(f + (I-S)v_n\right) + e_n. \tag{32}$$

THEOREM 12. *Let X be a Banach space with a uniformly convex dual, $S : X \to X$ a continuous and strongly accretive map such that $(I-S)$ has bounded range. Let $\{\alpha_n\} \in (0,1)$ satisfy* (3) *with $\beta_n \equiv 0$ and assume that the errors satisfy* (6). *If $u_0 = x_0 \in B$, then the following are equivalent:*

(i) *Mann iteration* (31) *converges to* $x^* \in F(T)$, *which is the solution of* $Sx = f$,

(ii) *Mann iteration with errors* (32) *converges to* $x^* \in F(T)$, *which is the solution of* $Sx = f$.

Theorem 13. [11] *Let* X *be a real Banach space with a uniformly convex dual,* $S : X \to X$ *a strongly accretive operator with* $(I - S)(X)$ *bounded. Let* $\{\alpha_n\} \in (0,1)$, $\{\beta_n\} \in [0,1)$ *satisfy* (3). *Then for* $u_0 = x_0 \in B$ *the following are equivalent:*

(i) *Mann iteration* (1) *with* $Tx := f + (I - S)x$ *converges to the solution of* $Sx = f$;

(ii) *Ishikawa iteration* (2) *with* $Tx := f + (I - S)x$ *converges to the solution of* $Sx = f$.

Theorems 11, 12, 13 and Remark 1 with $Tx = f + (I - S)x$ lead us to the following result.

Corollary 14. *Let* X *be a real Banach space with a uniformly convex dual,* $S : X \to X$ *a strongly accretive operator with* $(I - S)(X)$ *bounded. Let* $\{\alpha_n\} \in (0,1)$, $\{\beta_n\} \in [0,1)$ *satisfy* (3) *with errors* $\{p_n\}, \{q_n\}$ *satisfying* (8). *Then we have the equivalence between the convergences of* (1), (2), (7), (9), (10) *and* (5) *for* $Tx = f + (I - S)x$.

Consider the following operator equation

$$x + Sx = f,$$

where S is an accretive map and f is given. Consider iterations (2) and (7) with $Tx := f - Sx$:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n (f - Sy_n), \\ y_n &= (1 - \beta_n)x_n + \beta_n (f - Sx_n), \end{aligned} \tag{33}$$

and

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n (f - Sv_n) + p_n, \\ v_n &= (1 - \beta_n)u_n + \beta_n (f - Su_n) + q_n. \end{aligned} \tag{34}$$

Theorem 15. *Let* X *be a real Banach space with a uniformly convex dual, and* $S : X \to X$ *a continuous and accretive map with* $S(X)$ *bounded. Let* $\{\alpha_n\} \in (0,1)$, $\{\beta_n\} \in [0,1)$ *satisfy* (3), *with errors* $\{p_n\}, \{q_n\}$ *satisfying* (8). *If* $u_0 = x_0 \in B$, *then the following are equivalent:*

(i) *Ishikawa iteration* (33) *converges to a solution of* $x + Sx = f$

(ii) *Ishikawa iteration with errors* (34) *converges to a solution of* $x + Sx = f$.

Proof. In Theorem 6 set

$$Tx := f - Sx, \forall x \in X.$$

A fixed point for T will be a solution for $x+Sx=f$. The map S is an accretive map since, for all $j(x-y)\in J(x-y)$ we have

$$\langle Sx-Sy, j(x-y)\rangle \geq 0 \Rightarrow$$
$$\langle -Sx-(-Sy), j(x-y)\rangle \leq 0 \Rightarrow$$
$$\langle f-Sx-(f-Sy), j(x-y)\rangle \leq 0 \leq 0.5\left\|x-y\right\|^2.$$

Setting $k:=0.5\in(0,1)$ in (14), we get the definition of a strongly pseudocontractive map. □

In (33) and (34) set $\beta_n=0$, $q_n=0$, $\forall n\in\mathbb{N}$, to obtain the Mann iteration and the Mann iteration with errors:

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n\left(f-S\right)y_n\right), \tag{35}$$

and

$$u_{n+1}=(1-\alpha_n)u_n+\alpha_n\left(f-S\right)v_n\right)+e_n. \tag{36}$$

Theorem 16. *Let X be a Banach space with a uniformly convex dual, $S:X\to X$ a continuous and accretive map, let S be with a bounded range. Let $0<\alpha_n<1$ satisfy* (3) *with $\beta_n\equiv 0$. Suppose that the errors satisfy* (6). *If $u_0=x_0\in B$, then the following are equivalent:*

(i) *Mann iteration* (31) *converges to $x^*\in F(T)$, which is the solution of $x+Sx=f$,*
(ii) *Mann iteration with errors* (32) *converges to $x^*\in F(T)$, which is the solution of $x+Sx=f$.*

Theorem 17. [11] *Let X be a real Banach space with a uniformly convex dual, $S:X\to X$ an accretive operator with $S(X)$ bounded. Let $\{\alpha_n\}\in(0,1)$, $\{\beta_n\}\in[0,1)$ satisfy* (3). *Then for $u_0=x_0\in B$ the following assertions are equivalent:*

(i) *Mann iteration* (1) *with $Tx=f-Sx$ converges to the solution of $x+Sx=f$;*
(ii) *Ishikawa iteration* (2) *with $Tx=f-Sx$ converges to the solution of $x+Sx=f$.*

Theorems 15, 16, 17 and Remark 1 with $Tx=f-Sx$ lead us to the following result.

Corollary 18. *Let X be a real Banach space with a uniformly convex dual, $S:X\to X$ an accretive operator with $S(X)$ bounded. Let $\{\alpha_n\}\in(0,1)$, $\{\beta_n\}\in[0,1)$ satisfy* (3), *errors $\{p_n\},\{q_n\}$ satisfy* (8). *Then* (1), (2), (7), (9), (10)*and* (5) *are equivalent for $Tx=f-Sx$.*

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