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VECTOR SUBDIFFERENTIALS AND TANGENT CONES*

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Abstract. Following the Rockafellar's definition for the subdifferential of a real map we define a vector subdifferential using the normal cone to the epigraph of the function. For several kinds of normal cones we have different subdifferentials; we give properties, links between them, links with addapted directional derivatives and a genaralization for the Correa Joffré Thibault and for Zagrodny theorem from the real case.

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1. INTRODUCTION

It is well known that in nonsmooth analysis, the general definition given by Rockafellar [27] for the subdifferential of a lower semicontinuous function is expressed by means of the Clarke normal cone to its epigraph, denoted $N_{\mathrm{epi}\, f}$.

More precisely, for a proper, lower semicontinuous function $f : X \to \overline{\mathbb{R}}$, the subdifferential of f at $x_0 \in \mathrm{dom}\, f$ is given by

$$\partial f(x_0) = \{x^* \in X^* \mid (x^*, -1) \in N_{\mathrm{epi}\, f}(x_0, f(x_0))\}.$$

Since the Clarke normal cone is the polar of the Clarke tangent cone, this subdifferential is always a convex and closed set.

In [20], Mordukhovich introduced another normal cone which is not the polar of some tangent cone and generally, is not convex and closed. Using this normal cone, Mordukhovich defined a new type of nonconvex subdifferential which enjoys some interesting properties. Since the optimizations problems are often vectorial problems, many authors have tried to define and to study several types of vector subdifferentials.

The first ones to have considered such generalizations seem to be Raffin [25] and Valadier [37] who extended the convex subdifferential of Morreau and Rockafellar to vector functions.

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After that, several generalizations are given for the Clarke subdifferential using new kinds of directional derivatives, mostly for vector functions with values in a Banach lattice. We recall here the papers of A.G. Kusraev and S.S. Kutateladze [19], A.M. Rubinov [26], N. Papageorgiu [22], [23], L. Thibault [34], [35], J. Zowe [39], etc.

Another types of vector subdifferentials, (using the efficient points) were introduced by T. Tanino and Y. Sawaradgi [28] for the finite dimensional spaces, the so-called Pareto subdifferential, and were extended to the infinite dimensional spaces by A.B. Nemeth, G. Isac and V. Postolică [14].

As usual, another way to study a vectorial problem is the scalarization. By this way, some "mathematical objects" replacing the real subdifferential were introduced for example in [13].

In this paper, H.B. Urruty and L. Thibault define the set $\Gamma(f, x_0)$ for a locally Lipschitz mapping $f : X \to Y$, (where X is a separable Banach space and Y is a reflexive separable Banach space) like the convex closed maximal set from $\mathcal{L}(X, Y)$ for which $\partial y^* \circ f(x_0) = y^* \circ \Gamma(f, x_0),\ \forall y^* \in Y^*$.

This set can be seen as a generalization of the Clarke subdifferential given as the convex covering of the gradient limits. In the finite dimensional spaces, a such generalization was given by F.H. Clarke like the convex covering of the limits of jacobians matrices. If the dimension is greater than 1, the coincidence known for real convex function between the Clarke subdifferential and the convex subdifferential does not hold.

For the locally Lipschitz functions between two Banach spaces, in [16], the authors proposed another generalization for the subdifferential by the correspondence:

$$y^* \Longrightarrow \{x^* \in X^* \mid \langle x^*, h\rangle \le \langle y^*, f(x+h) - f(x)\rangle,\ \forall h \in X\}.$$

In [13] the authors provide the link between this notion and their generalization, the set $\Gamma(f, x_0)$.

This paper extends the Rockafellar definition of subdifferentials using the normal cones to the epigraph of the function, from real case to vectorial case, and generalizes some known results from real optimization with the aid of this new kinds of subdifferentials.

This study is structured in five parts: Introduction, Notations and preliminaries, Directional derivatives, Convex case and Nonconvex case, the last four being described shortly in what follows.

The second part presents the principal notions and results which will be used in the paper.

The third part introduces a type of directional derivatives using the tangent cones, and studies the link between the directional derivatives and the vector subdifferentials.

The fourth part is dedicated to the particular case in which the function is convex; we define a type of vector conjugate using the efficient points and we present the links with the known vector subdifferentials for convex functions.

The last part is dedicated to the Clarke and contingent subdifferentials. We give some generalizations for the real results like the Zagrodny mean theorem [38] and the Correa-Joffré-Thibault [7] result which provides the links between the monotonicity of the subdifferential and the convexity of the function.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper X, Z will be normed spaces, Z being endowed with a partial order induced by the cone Z_+ which is proper, closed, *convex* ($Z_+ + Z_+ \subset Z_+$), *pointed* ($Z_+ \cap -Z_+ = \{0\}$) and we will use the convention that $A + \emptyset = \lambda\emptyset = \emptyset$ for all $A \subset Y$, $\lambda \in \mathbb{R}$. We adjoin to Z an abstract maximal element denoted by $+\infty$ and an abstract minimal element denoted $-\infty$. We set $\overline{Z} = Z \cup \{+\infty\} \cup \{-\infty\}$. Infinity satisfies:

$$\lambda \cdot (\pm\infty) = \pm\infty \quad y + (\pm\infty) = \pm\infty,$$

for all $\lambda > 0$ and any y in Z.

For all $z, y \in Z$, we write

$$z \leq_{Z_+} y \iff y - z \in Z_+,$$
$$z \not<_{Z_+} y \iff y - z \notin Z_+ \setminus \{0\}.$$

If no confusion does hold, we denote simply $z \leq y$ and $z \not< y$.

We denote by $\mathcal{V}_Z(x_0)$ a fundamental system of neighbourhoods of x_0 for the given topology on Z. If the interior of the ordering cone Z_+ (denoted Int Z_+) is nonempty, then we shall consider for $+\infty$ a fundamental system of neighbourhoods consisting of the sets $(\varepsilon + Z_+) \cup \{+\infty\}$ with $\varepsilon \in \text{Int } Z_+$ and for $-\infty$ a fundamental system of neighbourhoods consisting of the sets $(-\varepsilon - Z_+) \cup \{-\infty\}$ with $\varepsilon \in \text{Int } Z_+$. As usualy, $\mathcal{L}(X, Z)$ stands for the set of linear continuous mappings from X to Z, Z^* is the topological *dual* of Z and Z_+^* is the *dual cone* of Z_+, defined by

$$Z_+^* = \Big\{ z^* \in Z^* \mid \forall z \in Z_+,\ z^*(z) \geq 0 \Big\},$$

while $Z_+^{\#}$ stands for the *quasi-interior* of Z_+^* and is given by

$$Z_+^{\#} = \Big\{ z^* \in Z_+^* \mid \forall z \in Z_+ \setminus \{0\}, z^*(z) > 0 \Big\}.$$

We adopt the convention that $z^*(\pm\infty) = \pm\infty$ for all $z^* \in Z_+^* \setminus \{0\}$. Recall that a convex cone Z_+ is normal if there exists a basis of neighbourhoods of the origin $\mathcal{V}(0)$ such that

$$(V - Z_+) \cap (V + Z_+) = V,\ \forall V \in \mathcal{V}(0);$$

The cone Z_+ is called Daniell if every decreasing net which has strong infimum is convergent to its infimum.

The principal sets of efficient points used in this paper are:

$$SUP\ A = \{z \in \overline{Z} \mid z \not< a,\ \forall z \in A \subset Z\};$$
$$SUP_1 A = \{z \in SUP\ A \mid \forall \varepsilon > 0,\ \exists a_\varepsilon \in A,\ a_\varepsilon > z - \varepsilon\};$$

$$INF\ A = \{z \in \overline{Z} \mid z \not> a,\ \forall z \in A \subset Z\};$$

$$INF_1 A = \{z \in INF\ A \mid \forall \varepsilon > 0,\ \exists a_\varepsilon \in A,\ a_\varepsilon < z + \varepsilon\};$$

if the interior of Z_+ is nonempty, we denote:

$$wSUP\ A = \{z \in \overline{Z} \mid z \not<_{\mathrm{Int}\ Z_+ \cup \{0\}} a,\ \forall z \in A \subset Z\};$$

$$wSUP_1 A = \{z \in wSUP\ A \mid \forall \varepsilon > 0,\ \exists a_\varepsilon \in A,\ a_\varepsilon > z - \varepsilon\};$$

$$wINF\ A = \{z \in \overline{Z} \mid z \not>_{\mathrm{Int}\ Z_+ \cup \{0\}} a,\ \forall z \in A \subset Z\};$$

$$wINF_1 A = \{z \in wINF\ A \mid \forall \varepsilon > 0,\ \exists a_\varepsilon \in A,\ a_\varepsilon < z + \varepsilon\};$$

(see [11] for more details). For a function $f : X \to \overline{Z}$ dom f (the domain of f) consists of the set of those points $x \in X$ for which $f(x) \neq +\infty$ and epi $f = \{(x, z) \in X \times Z \mid z \geq f(x)\}$ (the epigraph of f).

A map $f\colon X \to \overline{Z}$ is said to be *convex* (with respect to Z_+) if the epigraph of f is a convex set in $X \times Z$.

We will say that $f : X \to \overline{Z}$ is proper if $f(x) \neq -\infty$ for all $x \in X$ and there exists $x \in X$ with $f(x) \in Z$.

We recall here the definition of lower semicontinuity given in [33].

Definition 1. *A mapping $f : X \to Z$ is lower semicontinuous at x_0 if for all $V \in \mathcal{V}_Z(f(x_0))$ there exists $U \in \mathcal{U}_X(x_0)$ such that $f(U) \subset V + Z_+$.*

We say that f is lower semicontinuous if it is lower semicontinuous at each point of X.

If the interior of the cone Z_+ is nonempty, then we can extend this definition to the case of $f : X \to \overline{Z}$.

Definition 2. *A function $f : X \to \overline{Z}$ is lower semicontinuous at $x_0 \in X$ with $f(x_0) \in Z \cup \{+\infty\}$ if for all $V \in \mathcal{V}_Z(f(x_0))$ there exists $U \in \mathcal{U}_X(x_0)$ such that $f(U) \subset V + Z_+$. If $f(x_0) = -\infty$, then we consider that f is lower semicontinuous at x_0.*

We say that f is lower semicontinuous if it is lower semicontinuous at each point of X.

We will say that a function $f : X \to \overline{Z}$ is closed if epi f is a closed set in the product topology of $X \times Z$. If f is lower semicontinuous, then epi f is closed and thus f is closed. The converse statement is not true, see [33] for more details.

If f is convex and lower semicontinuous (closed), then $f(x) = -\infty$ for all $x \in \text{dom } f$ or $f(x) > -\infty$ for all $x \in X$.

For a multifunction $F : X \rightrightarrows Z$ we denote Dom $F = \{x \in X \mid\ F(x) \neq \emptyset\}$ (the domain of F), Gr $F = \{(x, z) \in X \times Z \mid\ z \in F(x)\}$ (the graph of F) and epi $F = \{(x, z) \in X \times Z \mid x \in \text{Dom } F,\ z \in F(x) + Z_+\}$ (the epigraph of F) .

We recall that a *multifunction* $F : X \rightrightarrows Z$ is *upper semicontinuous* at $x_0 \in \text{Dom } F$ if for all $V \in \mathcal{V}_Z(0)$, there exists $U \in \mathcal{V}_X(x_0)$ such that $F(x) \subset F(x_0) + V$ for all $x \in U$. The multifunction F is *lower semicontinuous* at x_0 if for any open subset $V \subset Z$ such that $V \cap F(x_0) \neq \emptyset$, there exists $U \in \mathcal{V}_X(x_0)$ such that $F(x) \cap V \neq \emptyset$ for all $x \in U$.

It is straightforward to verify the following lemmas.

LEMMA 3. *Let $f\colon X \to Z$ be a vector-valued mapping. The following assertions are equivalent:*

(1) *f is convex;*
(2) *for all z^* in the cone Z_+^*, the function $z^* \circ f$ is convex.*

LEMMA 4. *Let X and Z be two topological linear spaces and $f\colon X \to Z$ be a vector-valued mapping. If f is lower semicontinuous, then for each z^* in Z_+^*, the real function $z^* \circ f$ is lower semicontinuous.*

If the interior of Z_+ is nonempty, then the property remains valid for $f : X \to \overline{Z}$ and $z^* \in Z_+^* \setminus \{0\}$.

For a function $f : X \to \overline{Z}$ and $x_0 \in X$ such that $f(x_0) \in Z$, we shall use the following notations:

$$\partial_{\leq} f(x_0) = \{T \in \mathcal{L}(X,Z) \mid T(x - x_0) \leq f(x) - f(x_0),\ \forall x \in \operatorname{dom} f\}$$

(the usual Fenchel subdifferential used for instance in [39]).

$$\partial_{\not\geq} f(x_0) = \{T \in \mathcal{L}(X,Z) \mid\ T(x - x_0) \not\geq f(x) - f(x_0),\ \forall x \in \operatorname{dom} f\}$$

(the Pareto subdifferential used for instance in [14]).

If the interior of the cone Z_+ is nonempty, we denote:

$$w\partial_{\not\geq} f(x_0) = \{T \in \mathcal{L}(X,Z) \mid\ T(x - x_0) \not\geq_{\mathrm{Int}\ Z_+ \cup \{0\}} f(x) - f(x_0),\ \forall x \in \operatorname{dom} f\}$$

If $Z = \mathbb{R}$, we set by $\partial_{\leq} f(x)$ the usual convex subdifferential at x and by $\partial^{Cl} f(x)$ the usual Clarke subdifferential at x.

For a lower semicontinuous function $f : X \to \overline{Z}$ and $x \in \operatorname{dom} f$, $z^* \in Z_+^* \setminus \{0\}$ we define

$$\partial_{z^*} f(x) = \{T \in \mathcal{L}(X,Z) \mid (z^* \circ T, -z^*) \in N_{\mathrm{epi}\ f}(x, f(x))\} \tag{1}$$

and

$$\partial'_{z^*} f(x) = \{T \in \mathcal{L}(X,Z) \mid\ (z^* \circ T, -1) \in N_{\mathrm{epi}\ z^* \circ f}(x, z^* \circ f(x))\}, \tag{2}$$

where $N_{\mathrm{epi}\ f}(x, f(x))$ and $N_{\mathrm{epi}\ z^* \circ f}(x, z^* \circ f(x))$ denote the normal cone to the epigraph of f, and of $z^* \circ f$ respectively.

For the different types of normal cones we obtain the extension to the vector setting of the notions of subdifferentials for real valued functions.

Let recall from [cg:so] the following theorem which ensures the nonvacuity of the proximal infimum set (resp. for the proximal supremum set) for a large class of sets.

We will study the subdifferentials built with the aid of the polar cones for the cones $C_{\mathrm{epi}\ f}(x, f(x))$ (*the cone generated by the epigraph* of the function f), $T_{\mathrm{epi}\ f}(x, f(x))$ (*the tangent cone of Clarke to the epigraph* of the function

f), $K_{\text{epi } f}(x, f(x))$ (*the contangent cone or the Ursescu tangent cone to the epigraph* of the function f).

Recall that for a subset M of a normed space Y and for x belonging to the closure of M, the cone generated by the set M is

$$C_M(x) = \text{cl cone(M} - \text{x)},$$

(here cl cone K means the closure of the cone generated by K)
the Clarke tangent cone is the set

$$T_M(x) = \{v \in Y \mid \ \forall h_n \downarrow 0, \forall x_n \to_M x, \exists v_n \to v, \text{ s. t. } x_n + h_n v_n \in M, \forall n \in N\}$$

and the contangent cone is the set

$$K_M(x) = \{y \in Y \mid \forall t_k \to 0, \exists y_k \to y, \text{ such that, } x + t_k y_k \in M \forall k \in N\}.$$

Corresponding to the dual cone of each tangent cone described above and for $z^* \in Z_+^* \setminus \{0\}$, we denote the subdifferentials given in (1) and (2) by $\partial_{z^*}^F f(x)$, $\partial_{z^*}^{Cl} f(x)$, $\partial_{z^*}^U f(x)$, and by $\partial_{z^*}^{\prime F} f(x)$, $\partial_{z^*}^{\prime Cl} f(x)$, $\partial_{z^*}^{\prime U} f(x)$ respectively.

Let us remark that if $Z = \mathbb{R}$ the subdifferentials given in (1) coincide respectively with those given in (2) which in turn coincide respectively with the following real known subdifferentials: $\partial_{\leq} f(x)$-the Fenchel subdifferential, $\partial^C f(x)$-the Clarke subdifferential and $\partial^D f(x)$-the Dini subdifferential.

3. DIRECTIONAL DERIVATIVES

Unless otherwise stated, we will consider functions which are proper. Let $x \in$ dom f and we will denote by $TG_{\text{epi } f}(x, f(x))$ one of the tangent cones described above, $C_{\text{epi } f}(x, f(x))$, $T_{\text{epi } f}(x, f(x))$, $K_{\text{epi } f}(x, f(x))$. The polar cone of $TG_{\text{epi } f}(x, f(x))$ will be denoted $TG^\circ_{\text{epi } f}(x, f(x))$ and is defined as:

$$\begin{aligned} TG^\circ_{\text{epi } f}(x, f(x)) = \big\{(x^*, z^*) \in X^*, Z^* \mid x^*(u) + z^* v) \leq 0, \\ \forall (u, v) \in TG_{\text{epi } f}(x, f(x))\big\}. \end{aligned}$$

Note that if $(u, v) \in TG_{\text{epi}_f}(x, f(x))$, then $(u, v + z) \in TG_{\text{epi}_f}(x, f(x))$ for all $z \in Z_+$ and thus, if $(x^*, z^*) \in TG^\circ_{\text{epi } f}(x, f(x))$, then $z^* \in -Z_+^*$.

Since the normal cones in this case are polar cones of the tangent cones, we can consider a directional derivative Df defined as

$$\text{Gr } Df(x) = TG_{\text{epi } f}(x, f(x)).$$

Using this directional derivative we can express the subdifferentials as:

$$\begin{aligned} \partial_{z^*} f(x) &= \big\{T \in \mathcal{L}(X, Z) \mid z^* \circ T(u) \leq z^*(v), \\ &\qquad \forall v \in Df(x)(u), \ u \in Pr_X TG_{\text{epi } f}(x, f(x))\big\}, \\ \partial'_{z^*} f(x) &= \big\{T \in \mathcal{L}(X, Z) \mid z^* \circ T(u) \leq w, \\ &\qquad \forall w \in Dz^* \circ f(x)(u), \ u \in Pr_X TG_{\text{epi } z^* \circ f}(x, z^* \circ f(x))\big\}, \end{aligned}$$

where Pr_X means the projections on X.

In [1] we find the notion of directional derivative for a multifunction $F : X \rightrightarrows Z$ at $(x, y) \in \mathrm{Gr}\ F$ denoted by $\tilde{D}F(x, y)$ and given as $\mathrm{Gr}\tilde{D}\ F(x, y) = TG_{\mathrm{Gr}\ F}(x, y)$. For a function $f : X \to \overline{Z}$, if we consider the multifunction $\tilde{f} : X \rightrightarrows \bar{Z}$ given by

$$\tilde{f}(x) = \begin{cases} f(x) + Z_+ & if \qquad x \in \text{dom f} \\ Z & if \qquad f(x) = -\infty \\ \emptyset & if \qquad f(x) = +\infty \end{cases} \qquad (F)$$

then $D\ f(x) = \tilde{D}\ \tilde{f}(x, f(x))$.

By Rockafellar's terminology, let recall that a multifunction $F : X \rightrightarrows Z$ is a closed convex process if Gr F is a closed set and F is sublinear, i.e.

$$F(\lambda u) = \lambda F(u),\ \forall\lambda \geq 0, u \in X;$$

$$F(u) + F(v) \subseteq F(u + v), \forall u, v \in X.$$

If we denote by $D'f(x)$, $C^{\uparrow}f(x)$, $D^{\uparrow}f(x)$ the directional derivative corresponding to the cones $C_{\mathrm{epi}\ f}(x, f(x))$, $T_{\mathrm{epi}\ f}(x, f(x))$, and $K_{\mathrm{epi}\ f}(x, f(x))$ respectively, using the normal cones properties we find that $C^{\uparrow}f(x)$ is a closed convex process, $D^{\uparrow}f(x)$ is a closed processes and if f is a convex function, $D'f(x)$ is a closed, convex process, too.

Using the properties given for tangent cones in [1] (Theorem 2.2.6 and Proposition 6.2.3) we get the following remark:

Remark 1. If X, Z are Banach spaces and Dom $C^{\uparrow}f(x) = X$, then the multifunction $u \rightrightarrows C^{\uparrow}f(x)(u)$ is Lipschitz. If $f : X \to \overline{Z}$ is a Lipschitz mapping around a point $x \in$ dom f, then for a neighbourhood $\mathcal{U}$ of x, the multifunction

$$(y, u) \in \mathcal{U} \times X \rightrightarrows C^{\uparrow}z^* \circ f(y)(u)$$

is upper semicontinuous and the multifunction

$$u \rightrightarrows C^{\uparrow}z^* \circ f(x)(u)$$

is Lipschitz. □

The following proposition is a consequence of the lower semicontinuity of the multifunction

$$(x, y, u) \rightrightarrows \tilde{D}\tilde{f}(x, y)(u)$$

proved in Proposition 5.1.6. [1], restricted to the set $(U \cap Gr(f)) \times X$. Recall that a multifunction $F : X \rightrightarrows Z$ is *sleek* at $(x, y) \in \mathrm{Gr}(F)$ if the set-valued map

$$(x', y') \rightrightarrows K_{\mathrm{Gr}\ F}(x', y')$$

is lower semicontinuous at (x, y).

Proposition 5. *Let X, Z be Banach spaces and $f : X \to \overline{Z}$ be a map such that $\tilde{f}$ is sleek on a some neighbourhood U of $(x_0, y_0) \in Gr(\tilde{f})$. If the boundedness property*

$$\forall u \in X, \quad \sup_{(x,y)\in U\cap \mathrm{epi}\, f} \inf_{v\in \tilde{D}\tilde{f}(x,y)(u)} \|v\| \neq +\infty$$

holds true, then the set-valued map

$$(x, u) \Longrightarrow D^{\uparrow} f(x)(u)$$

is lower semicontinuous on $(\mathrm{Pr}_X(U \cap \mathrm{Gr}\ f) \times X$.

Proposition 6 and Corollaries 7, 8 are valid if the tangent cone $TG_{\mathrm{epi}\ f}(x, f(x))$ is either $C_{\mathrm{epi}\ f}(x, f(x))$ or $K_{\mathrm{epi}\ f}(x, f(x))$.

Proposition 6. *Let $f : X \to \overline{Z}$ be a lower semicontinuous function. For $z^* \in Z^*_+ \setminus \{0\}$ we have:*

$$(u, v) \in TG_{\mathrm{epi}\ f}(x, f(x)) \implies (u, z^*(v)) \in TG_{\mathrm{epi}\ z^*\circ f(x)}(x, z^* \circ f(x)).$$

Proof. The proof is obvious from the definition of the tangent cones. □

Thus we derive the following corollaries:

Corollary 7. *Let $f : X \to \overline{Z}$ be a proper, lower semicontinuous function $x_0 \in \mathrm{dom}\ f$ and $z^* \in Z^*_+ \setminus \{0\}$. Then for all $u \in \mathrm{Pr}_X TG_{\mathrm{epi}\ f}(x_0, f(x_0))$ we have:*

$$z^* \circ D\ f(x_0)(u) \subseteq D\ z^* \circ f(x_0)(u).$$

Corollary 8. *If $f : X \to \overline{Z}$ is a lower semicontinuous function, then for $z^* \in Z^*_+ \setminus \{0\}$ and $x_0 \in \mathrm{dom}\ f$ we have that*

$$\partial'_{z^*} f(x_0) \subseteq \partial_{z^*} f(x_0).$$

Proof. Using the Proposition 6, we deduce that

$$z^* \circ \partial'_{z^*} f(x_0) \subseteq z^* \circ \partial_{z^*} f(x_0).$$

Thus, if $T \in \partial'_{z^*} f(x_0)$, then $z^* \circ T \in z^* \circ \partial_{z^*} f(x_0)$ and so $(z^* \circ T, -z^*) \in TG^{\circ}_{\mathrm{epi}\ f}(x_0, f(x_0))$. We obtain the expected inclusion from the definitions of the subdifferentials. □

In fact, for the subdifferential corresponding to $C_{\mathrm{epi}\ f}(x_0, f(x_0))$ we have the equality

$$\partial^F_{z^*} f(x_0) = \partial'^F_{z^*} f(x_0). \tag{3}$$

Indeed, using the characterization of the normal cone $C^{\circ}_{\mathrm{epi}\ f}(x_0, f(x_0))$ we have:

$$\begin{aligned} T \in \partial^F_{z^*} f(x_0) &\Longleftrightarrow z^* \circ T(x - x_0) \leq z^* \circ f(x) - z^* \circ f(x_0),\ \forall x \in X \\ &\Longleftrightarrow z^* \circ T \in \partial_{\leq} z^* \circ f(x_0). \end{aligned}$$

It is a basic fact in convex analysis that

$$z^* \circ T \in \partial_{\leq} z^* \circ f(x_0) \Longleftrightarrow (z^* \circ T, -1) \in C^{\circ}_{\mathrm{epi}\ z^*\circ f}(x_0, f(x_0)) \Longleftrightarrow T \in \partial'_{z^*} f(x_0).$$

Thus, $\partial^F_{z^*} f(x_0) = \partial'^F_{z^*} f(x_0)$.

If $Z = \mathbb{R}$, we find that $\inf Df(x_0)$ is the known directional derivative:

$$\inf D' f(x_0)(u) = f'(x_0, u) = \inf_{t>0} \frac{f(x_0+tu)-f(x_0)}{t}$$

the convex directional derivative at x_0 in direction u;

$$\inf C^{\uparrow} f(x_0)(u) = f^{\uparrow}(x_0, u) = \sup_{\varepsilon>0} \liminf_{\substack{x\to x_0\\ f(x)\to f(x_0)\\ t\searrow 0}} \inf_{y\in B(u,\varepsilon)} \frac{f(x+ty)-f(x)}{t}.$$

the Rockafellar directional derivative at x_0 in direction u;

$$\inf D^{\uparrow} f(x_0)(u) = f'_D(x_0, u) = \liminf_{t\downarrow 0,\, y'\to u} \frac{f(x_0+ty')-f(x_0)}{t}$$

the inferior Dini derivative at x in direction u.

In what follows we present the specific properties for each cone.

4. CONVEX CASE

As the title of this section says, this part will be dedicated to the study of the vector subdifferentials for convex function. Since all the tangent cones to the epigraph of a convex lower semicontinuous function $f : X \to \overline{Z}$ at a point $(x_0, f(x_0)) \in \operatorname{epi} f$ coincide with

$$C_{\operatorname{epi} f}(x_0, f(x_0)) = \operatorname{cl}\operatorname{cone}(\mathrm{epi\ f} - (\mathrm{x_0}, \mathrm{f(x_0)})),$$

all the vector subdifferentials introduced in (1) and (2) will coincide with $\partial^F_{z^*} f(x_0) = \partial'^F_{z^*} f(x_0)$, as we have already seen in (3).

When $Z = \mathbb{R}$ and $g : X \to \overline{\mathbb{R}}$ is a proper, lower semicontinuous function then the two sets coincide with the usual real convex subdifferential given by

$$\partial_{\leq} g(x_0) = \{x^* \in X^* \mid x^*(x - x_0) \leq g(x) - g(x_0), \forall x \in X\}. \tag{6}$$

Observe that

$$\bigcap_{z^*\in Z^*_+\setminus\{0\}} \partial_{z^*} f(x_0) = \partial_{\leq} f(x_0), \tag{7}$$

and if $Z^{\#}_+ \neq \emptyset$, then

$$\bigcup_{z^*\in Z^{\#}_+} \partial_{z^*} f(x_0) \subseteq \partial_{\not>} f(x_0)$$

As consequences, we obtain:

Proposition 9. *Let $f : X \to \overline{Z}$ be a proper, lower semicontinuous function and $x_0 \in \operatorname{dom} f$.*

1. *$T \in \partial_{\leq} f(x_0)$ if and only if $Tu \leq v$, $\forall v \in D' f(x_0)(u)$, $\forall u \in \mathrm{Pr}_X C_{\operatorname{epi} f}(x_0, f(x_0))$;*
2. *If f is convex and the interior of Z_+ is nonempty then $T \in w\partial_{\not>} f(x_0)$ if and only if $Tu \not>_K v$, $\forall v \in D' f(x_0)(u)$, $\forall u \in \mathrm{Pr}_X C_{\operatorname{epi} f}(x_0, f(x_0))$.*

Proposition 10. *Let* $f : X \to \overline{Z}$ *be a proper, lower semicontinuous function. For* $x_0 \in \operatorname{dom} f$, $z^* \in Z_+^* \setminus \{0\}$, *the following equality does hold:*

$$z^* \circ \partial^F_{z^*} f(x_0) = \partial_{\leq} z^* \circ f(x_0). \tag{8}$$

Proof. The inclusion $z^* \circ \partial^F_{z^*} f(x_0) \subseteq \partial_{\leq} z^* \circ f(x_0)$ is obvious.

Now, take $x^* \in \partial_{\leq} z^* \circ f(x_0)$, and $T \in \mathcal{L}(X, Z)$ such that $x^*(x) = z^* \circ T(x)$. Thus, $z^* \circ T \in \partial_{\leq} z^* \circ f(x_0)$, which amounts to saying that $T \in \partial^F_{z^*} f(x_0)$. Indeed, we can take the operator T given by $Tx = x^*(x)e$ with e an element of Z such that $z^*(e) = 1$. Since $x^* \in X^*$, $z^* \circ T = x^*$ and $T \in \mathcal{L}(X, Z)$, we get $z^* \circ \partial^F_{z^*} f(x_0) \supseteq \partial_{\leq} z^* \circ f(x_0)$ and thus the equality follows. □

Following the line of the scalar case, we define now a type of vectorial conjugate which will be used for the characterization of the subdifferentiability.

Definition 11. *Let* $f : X \to \overline{Z}$ *be a proper, lower semicontinuous function and* $T \in \mathcal{L}(X, Z)$. *We denote by* $f^*(T)$ *the vectorial conjugate of* f *at* T *given by*

$$f^*(T) = SUP_1\{Tx - f(x),\ x \in \operatorname{dom} f\}.$$

If $\operatorname{Int} Z_+ \neq \emptyset$, *we define the weak vectorial conjugate of* F *at* T *denoted by* $wf^*(T)$ *as*

$$wf^*(T) = wSUP_1\{Tx - f(x),\ x \in \operatorname{dom} f\}.$$

Under the hypothesis of Theorem 1.3 [30] we observe that $wf^*(T) = +\infty$ or $wf^*(T)$ is a nonempty set of Z.

Proposition 12. *Let* $f : X \to \overline{Z}$ *be a proper, lower semicontinuous function and* $x_0 \in \operatorname{dom} f$.

(1) *If* $z^* \in Z_+^* \setminus \{0\}$ *and* $T \in \partial^F_{z^*} f(x_0)$, *then*

$$z^*(Tx_0 - f(x_0)) \geq \max z^* \circ f^*(T).$$

(2) *If* $T \in \partial_{\leq} f(x_0)$, *then*

$$Tx_0 - f(x_0) = f^*(T).$$

Proof. 1. Let $z^* \in Z_+^* \setminus \{0\}$ and $T \in \partial^F_{z^*} f(x_0)$. Thus,

$$z^*(Tx_0 - f(x_0)) = \max\{z^*(Tx - f(x)) \mid x \in \operatorname{dom} f\}.$$

Let $\alpha \in f^*(x_0)$; for all $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that

$$T(x_\varepsilon) - f(x_\varepsilon) > \alpha - \varepsilon.$$

Since $z^* \in Z_+^* \setminus \{0\}$, this yields:

$$\max(z^* \circ f^*(T)) \leq \max\{z^*(Tx - f(x)) \mid x \in \operatorname{dom} f\}.$$

Finally, we obtain that

$$z^*(Tx_0 - f(x_0)) \geq \max z^* \circ f^*(T).$$

2. If $T \in \partial_{\leq} f(x_0)$ then $Tx_0 - f(x_0) \geq Tx - f(x)$ for all $x \in \operatorname{dom} f$ and thus,

$$Tx_0 - f(x_0) \in f^*(T).$$

Now, for $\alpha \in f^*(T)$ and $\varepsilon > 0$ we find $x_\varepsilon \in X$ such that

$$\alpha - \varepsilon < Tx_\varepsilon - f(x_\varepsilon) < Tx_0 - f(x_0).$$

This implies that $\alpha \le Tx_0 - f(x_0)$ and therefore we have $Tx_0 - f(x_0) = f^*(T)$, as desired. ☐

Proposition 13. *Let $f : X \to \overline{Z}$ be a proper, lower semicontinuous function, $x_0 \in \operatorname{dom} f$, $T \in \mathcal{L}(X, Z)$, and suppose that the interior of the cone Z_+ is nonempty.*

(1) *If f is convex, then there exists $z^* \in Z_+^* \setminus \{0\}$ such that $T \in \partial^F_{z^*} f(x_0)$ if and only if*

$$Tx_0 - f(x_0) \in wf^*(T);$$

(2) *If in addition Z_+ is normal and Daniell, $T \in w\partial_{\le} f(x_0)$ if and only if*

$$T(x_0) - f(x_0) = wf^*(T). \tag{9}$$

Proof. 1. Firstly, let us suppose that there exists $z^* \in Z_+^*$, $z^* \neq 0$ such that $T \in \partial^F_{z^*} f(x_0)$. Thus,

$$z^*(Tx_0 - f(x_0)) = \max\{z^*(Tx - f(x)) \mid \ x \in X\}.$$

This yields that for each $x \in \operatorname{dom} f$

$$Tx_0 - f(x_0) \notin Tx - f(x) - \operatorname{Int}\ Z_+.$$

Otherwise, it would exist $z \in \operatorname{Int}\ Z_+$ such that $z^*(z) = 0$ which would imply that $z^* = 0$. Thus, the relation (9) gives that $Tx_0 - f(x_0) \in wf^*(T)$.

Let us consider now that $Tx_0 - f(x_0) \in wf^*(T)$. Thus,

$$Tx_0 - f(x_0) \notin \{Tx - f(x) - \operatorname{Int} Z_+ \mid \ x \in \operatorname{dom} f\}.$$

Since f is convex and the cone Z_+ is convex, according to the Hahn-Banach separation theorem we can select $z^* \in Z_+^* \setminus \{0\}$ such that

$$z^*(Tx_0 - f(x_0)) \ge z^*(Tx - f(x)),\ \forall x \in \operatorname{dom} f.$$

Obviously, if $f(x) = +\infty$, then the previous inequality does hold. The last relation says exactly that $T \in \partial^F_{z^*} f(x_0)$ and thus the assertion is proved.

2. The previous proposition yields that the condition is necessary and $wf^*(T) \neq +\infty$. Suppose that (9) does hold. From Theorem 1.3 [30] we know that for all $x \in \operatorname{dom} f$, there exists some $\alpha \in wf^*(T)$ such that

$$Tx - f(x) - \alpha \in -(\operatorname{Int} Z_+ \cup \{0\}).$$

Since $wf^*(T) = Tx_0 - f(x_0)$, we obtain that for all $x \in \operatorname{dom} f$

$$Tx - f(x) - (Tx_0 - f(x_0)) \in -(\operatorname{Int} Z_+ \cup \{0\}).$$

This amounts to saying that $T \in w\partial_{\le} f(x_0)$. ☐

Proposition 14. *Let* $f : X \to \overline{Z}$ *be a proper, convex, lower semicontinuous function and suppose that* $\operatorname{Int} Z_+ \neq \emptyset$. *Then, for* $x_0 \in \operatorname{dom} f$,

$$w\partial_{\not>} f(x_0) = \bigcup_{z^* \in Z_+^* \setminus \{0\}} \partial^F_{z^*} f(x_0).$$

Proof. Let $T \in \bigcup_{z^* \in Z_+^* \setminus \{0\}} \partial^F_{z^*} f(x_0)$. This means that there exists $z^* \in Z_+^* \setminus \{0\}$ such that:

$$z^*(Tx - Tx_0) \leq z^*(f(x) - f(x_0)),\ \forall x \in \operatorname{dom} f.$$

Since the interior of the cone Z_+ is nonempty we have

$$Tx - Tx_0 \not>_K f(x) - f(x_0),\ \forall x \in \operatorname{dom} f. \tag{10}$$

This amounts to saying that $T \in w\partial_{\not>} f(x_0)$. Suppose now that $T \in w\partial_{\not>} f(x_0)$ and thus (10) does hold. Since f is convex, according to the Hahn-Banach separation theorem applied for the sets $\{Tx_0 - f(x_0)\}$ and $\{Tx - f(x) - (\operatorname{Int} Z_+ \cup \{0\}) \mid x \in \operatorname{dom} f\}$, we can select $z^* \in Z_+^* \setminus \{0\}$ such that

$$z^*(Tx - Tx_0) \leq z^*(f(x) - f(x_0)),\ \forall x \in \operatorname{dom} f. \tag{11}$$

Relation (11) says exactly that $T \in \partial^F_{z^*} f(x_0)$ and the proof is complete. □

The following results intend to establish necessary and sufficient conditions for the equality from $\partial_{\leq} f(x_0)$ and $\partial_{\not>} f(x_0)$.

Example 1. If the order is not total, then we can easily give an example of a mapping $f \colon X \to Z^{\cdot}$ such that $\partial_{\leq} f(0) = \emptyset$ and $\partial_{\not>} f(0) \neq \emptyset$. For this, pick $T \in \mathcal{L}(X, Z)$ and define $f(x) = T(x) + \alpha$ (where $\alpha \in Z \setminus (Z_+ \cup -Z_+)$) for all $x \neq 0$ and $f(0) = 0$. Then, $T \in \partial_{\not>} f(0)$ while $\partial_{\leq} f(0) = \emptyset$. □

Example 2. Obviously, the inclusion $\partial_{\leq} f(x_0) \subseteq \partial_{\not>} f(x_0)$ holds for all $x_0 \in \operatorname{dom} f$. Let X and Z be two Hausdorff locally convex spaces and $f \colon X \to Z$ be a constant function. Then

$$\partial_{\leq} f(x_0) = \{0\} \text{ and } \partial_{\not>} f(x_0) = \{T \in \mathcal{L}(X, Z) \,|\, T(x) \not> 0,\ T(x) \not< 0,\ \forall x \in X\}.$$

If the order is not total, then for $x^* \in X^* \setminus \{0\}$ and $\alpha \in Z \setminus (Z_+ \cup -Z_+)$, the operator $T = \alpha x^*$ belongs to $\partial_{\not>} f(x_0)$ and $T \neq 0$. Thus, $\partial_{\leq} f(x_0) \subset \partial_{\not>} f(x_0)$. □

Proposition 15. *Let* $f \colon X \to Z$ *be a mapping such that* $\partial_{\leq} f(x_0) \neq \emptyset$. *Then* $\partial_{\leq} f(x_0) = \partial_{\not>} f(x_0)$ *if and only if "*$\leq$*" is a total ordering relation.*

Proof. Suppose that "$\leq$" is not a total order. Thus Z has an algebraic dimension larger or equal to 2 and there exists $z \in Z \setminus (Z_+ \cup Z_-)$. The operator $T_1 \in \mathcal{L}(X, Z)$ given by $T_1(x) = x^*(x) z$ with $z \in Z \setminus (Z_+ \cup -Z_+)$ and $x^* \in X^* \setminus \{0\}$, has the property that

$$T_1 \neq 0 \text{ and } \forall x \in X,\ T_1(x) \not> 0,\ T_1(x) \not< 0.$$

Now, for T in $\partial_{\leq} f(x_0)$ we have:

$$(T+\lambda T_1)(x-x_0) \not> f(x)-f(x_0),\ \forall x \in X,\ \lambda \in \mathbb{R}_+.$$

Therefore, $T+\lambda T_1$ belongs to $\partial_{\not>} f(x_0)$ and since the subdifferentials are equal at x_0, we have that $T+\lambda T_1$ belongs to $\partial_{\leq} f(x_0)$ for all $\lambda > 0$. This means that

$$(T+\lambda T_1)(x-x_0) \leq f(x)-f(x_0),\ \forall x \in X,\ \forall \lambda > 0.$$

In particular, for an arbitrarily chosen x we have

$$T_1(x-x_0) \leq (f(x)-f(x_0)-T(x-x_0))/\lambda,\ \forall \lambda > 0.$$

This yields $T_1(x-x_0) \leq 0$ for all x in X, which implies that $T_1 = 0$, contradiction.

For the sufficiency, it is obvious that if the order is total, then the two subdifferentials coincide. □

The last result is a generalization of the similar result well known for the real valued mappings.

Proposition 16. *Let X and Z be two real reflexive Banach spaces and suppose that the ordering cone Z_+ has a weakly compact base and has nonempty interior. Let $f, g\colon X \to Z$ be two convex continuous functions. The following assumptions are equivalent:*

(1) $\partial_{\leq} f(x) \subseteq \partial_{\leq} g(x)$, *for all x in X;*
(2) $\partial_{\leq} f(x) = \partial_{\leq} g(x)$, *for all x in X;*
(3) $f(x) = g(x) + k$, *for all x in X.*

Proof. From Lemma 2.26 of [18] (p. 52),

$$\partial_{\leq}(z^* \circ f)(x) \ = \ z^* \circ \partial_{\leq} f(x) \subseteq z^* \circ \partial_{\leq} g(x) \ = \ \partial_{\leq}(z^* \circ g)(x).$$

Thus, $\partial_{\leq}(z^* \circ f)(x) \subseteq \partial_{\leq}(z^* \circ g)(x)$ which yields

$$z^* \circ f(x) = z^* \circ g(x) + k(z^*),\ \forall\, z^* \in Z_+^*,\ \forall\, x \in X.$$

In particular, $z^* \circ f(x) - z^* \circ g(x) = z^* \circ f(0) - z^* \circ g(0)$, $\forall z^* \in Z_+^*$. Since $\operatorname{Int} Z_+^* \neq \emptyset$, the cone Z_+^* generates Z^*, *i.e.* $Z^* = Z_+^* - Z_+^*$ and thus the previous relation holds for all $z^* \in Z^*$.

As a result,

$$f(x) - g(x) = f(0) - g(0) = k,$$

and the equivalences follow obviously. □

5. NONCONVEX CASE

This section is dedicated to the study of vector subdifferentials for nonconvex functions. As we have already seen in the Section 2, we can define such subdifferentials using the tangent cones, like the Clarke tangent cone and the contangent cone.

We remark that if $\operatorname{dom} f = X$, we can define the sets from (2) for $z^* \in Z^* \setminus \{0\}$.

PROPOSITION 17. *Let $f : X \to \overline{Z}$ be a lower semicontinuous function and $z^* \in Z_+^* \setminus \{0\}$. Then, for $x_0 \in \mathrm{dom}\, f$*

$$\partial_{z^*}^F f(x_0) \subseteq \partial_{z^*}^U f(x_0) \subseteq \partial_{z^*}^{Cl} f(x_0)$$

and

$$\partial_{z^*}^F f(x_0) \subseteq \partial_{z^*}'^U f(x_0) \subseteq \partial_{z^*}'^{Cl} f(x_0).$$

If f is a convex proper lower semicontinuous function, then equality holds in these inclusions.

Proof. The following inclusions between tangent cones

$$C_{\mathrm{epi}\, f}(x_0, f(x_0)) \supseteq K_{\mathrm{epi}\, f}(x_0, f(x_0)) \supseteq T_{\mathrm{epi}\, f}(x_0, f(x_0)),$$

yield

$$z^* \circ \partial_{z^*}^F f(x_0) \subseteq z^* \circ \partial_{z^*}^U f(x_0) \subseteq z^* \circ \partial_{z^*}^{Cl} f(x_0).$$

Thus,

$$\partial_{z^*}^F f(x_0) \subseteq \partial_{z^*}^U f(x_0) + M_{z^*} \subseteq \partial_{z^*}^{Cl} f(x_0) + M_{z^*},$$

where $M_{z^*} = \{T \in \mathcal{L}(X, Z) \mid \ z^* \circ T = 0\}$.

Since $\partial_{z^*}^U f(x_0) + M_{z^*} = \partial_{z^*}^U f(x_0)$ and $\partial_{z^*}^{Cl} f(x_0) + M_{z^*} = \partial_{z^*}^{Cl} f(x_0)$, we derive that

$$\partial_{z^*}^F f(x_0) \subseteq \partial_{z^*}^U f(x_0) \subseteq \partial_{z^*}^{Cl} f(x_0).$$

Similarly we obtain

$$\partial_{z^*}^F f(x_0) \subseteq \partial_{z^*}'^U f(x_0) \subseteq \partial_{z^*}'^{Cl} f(x_0).$$

If f is convex, then all concepts of tangent cones coincide and thus the equality holds in these inclusions. □

PROPOSITION 18. *If X, Z are finite dimensional spaces, $f : X \to \overline{Z}$ is a lower semicontinuous function such that $\partial_{z^*}^U f(y) \neq \emptyset$ for $y \in V \in \mathcal{V}(x)$, $x \in$* dom f *and $z^* \in Z_+^* \setminus \{0\}$, then*

$$\operatorname*{Limsup}_{y \to x,\ f(y) \to f(x)} \partial_{z^*}^U f(y) \subset \partial_{z^*}^{Cl} f(x).$$

Proof. Under our assumption, Theorem 4.4.3 [1] shows that

$$(T_{\mathrm{epi}\, f}(x, f(x)))^\circ = \overline{\mathrm{co}} \operatorname*{Limsup}_{(y,u) \to_{\mathrm{epi}\, f} (x, f(x))} (K_{\mathrm{epi}\, f}(y, u))^\circ.$$

Thus, if $U \in \operatorname*{Limsup}_{y \to x,\ f(y) \to f(x)} \partial_{z^*}^U f(y)$, then

$$(z^* \circ U, -z^*) \in \operatorname*{Limsup}_{(y,u) \to_{\mathrm{epi}\, f} (x, f(x))} (K_{\mathrm{epi}\, f}(y, u))^\circ \subset (T_{\mathrm{epi}\, f}(x, f(x)))^\circ.$$

Thus, $U \in \partial_{z^*}^{Cl} f(x)$ and finally,

$$\operatorname*{Limsup}_{y \to x,\ f(y) \to f(x)} \partial_{z^*}^U f(y) \subset \partial_{z^*}^{Cl} f(x). \qquad \square$$

The following result extends to the vector setting the theorem of Correa-Jofré–Thibault, which expresses the convexity for a function in terms of the monotonicity of its Clarke subdifferential.

We recall now the definition of a monotone operator.

Definition 19. *We say that the operator* $\Delta : D \subset X \rightrightarrows \mathcal{L}(X, Z)$ *is monotone if for all* $x, y \in D$ *and* $T \in \Delta(x)$, $U \in \Delta(y)$, *we have*

$$(T - U)(x - y) \geq 0.$$

We say that Δ *is a maximal monotone operator if it is maximal in the set of the monotone operators* $V : D \subset X \rightrightarrows \mathcal{L}(X, Z)$.

Proposition 20. *Let* X *and* Z *be reflexive Banach spaces and let* Z *be ordered by a closed convex pointed and proper cone* Z_+ *which has a compact base. Let* $f : X \to Z$ *be a continuous mapping. The following assertions are equivalent:*

(1) f *is convex;*

(2) $z^* \circ \partial_{\leq} f(x) = \partial^{Cl}\Big(z^* \circ f\Big)(x)$, $\forall z^* \in Z_+^*$, $\forall x \in X$.

Proof. "1. $\Longrightarrow$ 2." is the Jahn's result [18].

Conversely, if the relation $z^* \circ \partial_{\leq} f(x) = \partial^{Cl}(z^* \circ f)(x)$ does hold for all $z^* \in Z_+^*$ and $x \in X$, then $\partial^{Cl}(z^* \circ f)$ is obviously a monotone operator since $\partial_{\leq} f$ is monotone and $z^* \in Z_+^*$. Combining the result of Correa *et al.* ([7], Theorem 3.8) and Lemma 3, we obtain that f is convex, establishing the result. □

Corollary 21. *By the same hypothesis like in Proposition 20 we obtain that* f *is convex if and only if* $\partial'^{Cl}_{z^*} f(x) = \partial_{\leq} f(x) + M_{z^*}$ *for all* $z^* \in Z_+^* \setminus \{0\}$ *and* $x \in X$.

If $Z = \mathbb{R}$, Proposition 20 says that if X is a reflexive space and f is a continuous function, then f is convex if and only if $\partial_{\leq} f(x) = \partial^{Cl} f(x)$. We recover this result from [7], Proposition 2.3 and Theorem 3.8.

Corollary 22. *By the same hypothesis like in Proposition 20 we derive that* f *is convex if and only if*

$$\partial^F_{z^*} f(x) = \partial'^{Cl}_{z^*} f(x) \; \forall x \in X, \; z^* \in Z_+^* \setminus \{0\}. \tag{16}$$

Proof. Obviously, if f is convex, the equality follows in (16).

Now, if the relation (16) does hold we derive that $\partial^{Cl} z^* \circ f$ is a monotone operator for all $z^* \in Z_+^* \setminus \{0\}$ and using the Correa-Jofré-Thibault result (Theorem 3.8 [7]) we obtain the equivalence. □

Proposition 23. *Let* $f : X \to \overline{Z}$ *be a proper function such that* $z^* \circ f$ *is a lower semicontinuous function for all* $z^* \in Z_+^* \setminus \{0\}$. *If* $\partial^{Cl}_{z^*} f$ *is monotone, then* $z^* \circ f$ *is a convex function for all* $z^* \in Z_+^* \setminus \{0\}$. *Conversely, if* f *is convex, then* $\bigcap\limits_{z^* \in Z_+^* \setminus \{0\}} \partial^{Cl}_{z^*} f$ *is a monotone operator.*

Proof. Under this hypothesis, $T_{\mathrm{epi}\, f}(x_0, f(x_0)) = K_{\mathrm{epi}\, f}(x_0, f(x_0)) = C_{\mathrm{epi}\, f}(x_0, f(x_0))$ and $\bigcap\limits_{z^*\in Z_+^*\setminus\{0\}} \partial_{z^*}'^{Cl} f = \partial_{\leq} f$ which is obviously monotone.

Now, if $\partial_{z^*}'^{Cl} f$ is monotone, then $z^* \circ \partial_{z^*}'^{Cl} f$ is monotone and thus $\partial^{Cl} z^* \circ f$ is also monotone. From Lemma 3 and [7] we obtain that $z^* \circ f$ is convex. □

We present now a vectorial version of the Zagrodny mean value Theorem for the nonsmooth functions, whose proof follows the Thibault's one adapted to the vectorial case.

THEOREM 24. *Let $(X, \|\cdot\|)$ be a Banach space and $(Z, \|\cdot\|)$ be a separable normed space, ordered by a pointed closed convex cone Z_+. Let $f : X \to Z \cup \{+\infty\}$ be a function such that $z^* \circ f$ is lower smicontinuous for all $z^* \in Z_+^* \setminus \{0\}$. Given two distinct points a, b in the domain of f, $z^* \in Z_+^{\#} \setminus \{0\}$ and $\varepsilon \in Z_+ \setminus \{0\}$, there exists a sequence $(x_n)_{n\in\mathbf{N}}$ converging to c in $[a, b)$ such that $\lim\limits_{n\to+\infty} z^* \circ f(x_n) = z^* \circ f(c)$, and there exist n_0 and $T_n \in \partial_{z^*}'^{Cl} f(x_n)$ such that for each $n \geq n_0$,*

(1) $T_n(b - x_n)/\|b - x_n\| \geq -\varepsilon/n + (f(b) - f(a))/\|b - a\|$,
(2) $T_n(b - a)/\|b - a\| \geq -\varepsilon/n + (f(b) - f(a))/\|b - a\|$,
(3) $\|b - a\|\, z^*(f(c) - f(a)) \leq \|c - a\|\, z^*(f(b) - f(a))$.

Proof. We may suppose that $f(b) = f(a)$ (otherwise we just consider the function

$$g(x) = \|b - a\|\, f(x) + \|b - x\|\, (f(b) - f(a))$$

which satisfies our condition). Since $(Z, \|\cdot\|)$ is a separable normed space, the quasi-interior $Z_+^{\#} \subseteq Z_+^* \setminus \{0\}$ is nonempty. From the lower semicontinuity of $z^* \circ f$ (with $z^* \in Z_+^* \setminus \{0\}$), we can find a point $c \in [a, b)$ which is a minimum point for $z^* \circ f$ on $[a, b]$ and $r > 0$ such that $z^* \circ f$ is lower bounded on $U = [a, b] + rB$ (let γ be this lower bound). We consider $f_U(x) = f(x)$ for $x \in U$ and $f_U(x) = +\infty$, else. For each n, let $r_n \in (0, r)$ such that for $x \in [a, b] + r_n B$

$$z^* \circ f(x) \geq z^* \circ f(c) - \tfrac{z^*(\varepsilon)}{n^2}$$

and let $t_n \geq n$ and u in $Z_+ \setminus \{0\}$ such that $z^*(u) = 1$ and

$$\gamma + t_n z^*(u) r_n \geq z^* \circ f(c) - \tfrac{z^*(\varepsilon)}{n^2}.$$

We obtain

$$z^* \circ f(c) \leq \inf_{x\in X} z^* \circ f_U(x) + t_n z^*(u) d_{[a,b]}(x) + \tfrac{z^*(\varepsilon)}{n^2}$$

(here $d_{[a,b]}(\cdot)$ is the distance function to $[a, b]$ and B is the closed unit ball of X). We can apply the Ekeland variational principle for the lower semicontinuous function

$$F_n = z^* \circ (f_U + t_n u d_{[a,b]}),$$

and we get $x_n \in X$ such that

i) $\|c - x_n\| \leq 1/n$,

ii) $F_n(x_n) \le F_n(c) = z^* \circ f(c)$,

iii) $\forall x \in X,\ F_n(x_n) \le F_n(x) + \frac{1}{n} z^*(\varepsilon) \|x - x_n\|$.

From i), we may suppose that x_n belongs to the interior of U. Now, iii) gives that

$$0 \in \partial^{Cl}(F_n + \tfrac{z^*(\varepsilon)}{n} \|\cdot - x_n\|)(x_n).$$

Using the properties of the Clarke subdifferential we derive:

$$0 \in \partial^{Cl} z^* \circ f(x_n) + \partial^{Cl}(z^* \circ \tfrac{\varepsilon}{n} \|\cdot - x_n\|)(x_n) + \partial^{Cl} t_n z^*(u) d_{[a,b]}(x_n).$$

Since the distance function and the norm are convex and continuous, we get

$$0 \in \partial^{Cl} z^* \circ f(x_n) + \partial_{\le}(z^* \circ \tfrac{\varepsilon}{n} \|\cdot - x_n\|)(x_n) + \partial_{\le} t_n z^*(u) d_{[a,b]}(x_n).$$

Thus, there exist $u_n^* \in \partial^{Cl} z^* \circ f(x_n)$, $v_n^* \in \partial_{\le} d_{[a,b]}(x_n)$ and $b_n^* \in B^*$ (where B^* is the unit ball of the X^*) such that

$$-u_n^* = z^*(t_n u v_n^* + \tfrac{\varepsilon}{n} b_n^*).$$

(Here $\varepsilon b_n^*,\ u v_n^* \in \mathcal{L}(X, Z)$ are defined by $(\varepsilon b_n^*)(x) = b_n^*(x)\varepsilon$ and $(u v_n^*)(x) = v_n^*(x)\varepsilon$ for all $x \in X$, respectively.)

We denote $T_n = -t_n u v_n^* - \varepsilon/n b_n^*$ and using the definition of $\partial'^{Cl}_{z^*} f$ we have that $T_n \in \partial'^{Cl}_{z^*} f$.

If we choose $y_n \in [a, b]$ with $\|x_n - y_n\| = d_{[a,b]}(x_n)$ we have from i) that $y_n \to c$ and moreover,

$$v_n^*(b - x_n) \le d_{[a,b]}(b) - d_{[a,b]}(x_n) = -d_{[a,b]}(x_n) \le 0.$$

Since $\|v_n^*\| \le 1$,

$$\begin{aligned} v_n^*(b - y_n) &= v_n^*(b - x_n) + v_n^*(x_n - y_n) \\ &\le d_{[a,b]}(b) - d_{[a,b]}(x_n) + \|v_n^*\| \, \|x_n - y_n\| \\ &\le -d_{[a,b]}(x_n) + d_{[a,b]}(x_n) = 0. \end{aligned}$$

As $y_n \to c \ne b$, there exists n_0 such that for $n \ge n_0$ we have $y_n \in [a, b)$ and the precedent inequalities imply that

$$v_n^*(b - a) \le 0.$$

Since $b_n^* \in B^*$, we have

$$\begin{aligned} b_n^*(b - x_n) &\le \|b - x_n\|, \\ b_n^*(b - a) &\le \|b - a\|. \end{aligned}$$

We derive for each n,

$$\begin{aligned} T_n(b - x_n) &\ge \tfrac{-\varepsilon}{n} \|b - x_n\|, \\ T_n(b - a) &\ge \tfrac{-\varepsilon}{n} \|b - a\|. \end{aligned}$$

The third relation follows from iii). □

Remark 2. Following Lemma 4, the lower semicontinuity of $z^* \circ f$ for each $z^* \in Z_+^* \setminus \{0\}$ is assured if $\operatorname{Int} Z_+ \ne \emptyset$. □

Remark 3. If $Z = \mathbb{R}$, we retrieve the real Zagrodny mean theorem by passing to the "liminf" in the relations 1,2 and by taking the subsequences. $\square$

Remark 4. The conclusion from the precedent theorem rests valid if Int $Z_+ \neq \emptyset$, $\varepsilon \in \mathrm{Int}\, Z_+$ and $z^* \in Z_+^*$. $\square$

Using this theorem, we can give a characterization of the monotonicity of $\partial'^{Cl}_{z^*} f$.

Corollary 25. *Let $(X, \|\cdot\|)$ be a Banach space and $(Z, \|\cdot\|)$ be a separable normed space, ordered by a pointed closed convex cone Z_+ with nonempty interior, and let f be a lower semicontinuous function, $f : X \to Z \cup \{+\infty\}$. Then, for $z^* \in Z_+^* \setminus \{0\}$, $\partial'^{Cl}_{z^*} f$ is monotone if and only if $\partial'^{Cl}_{z^*} f(x) = \partial_{\leq} f(x)$, for all $x \in X$.*

Proof. Obviously, if $\partial'^{Cl}_{z^*} f(x) = \partial_{\leq} f(x)$, for all $x \in X$, then $\partial'^{Cl}_{z^*} f$ is monotone. Now, if $\partial'^{Cl}_{z^*} f(x) = \emptyset$, then the equality follows using the fact that $\partial'^{Cl}_{z^*} f(x) \supseteq \partial_{\leq} f(x)$.

Let $x \in X$ such that $\partial'^{Cl}_{z^*} f(x) \neq \emptyset$. Then, $x \in \mathrm{dom}\, f$ and following the precedent remark we can apply the vectorial version of the Zagrodny theorem for each d such that $x + d \in \mathrm{dom}\, f$, $\varepsilon \in \mathrm{Int}\, Z_+$. Thus, there exist $v_k \to d$, $t_k \to t \in (0, 1]$ and $T_k \in \partial'^{Cl}_{z^*} f(x + t_k v_k)$ such that

$$\tfrac{f(x+d)-f(x)}{\|d\|} \geq \tfrac{T_k(v_k)}{\|v_k\|} - \tfrac{\varepsilon}{k}.$$

Using the monotonicity of $\partial'^{Cl}_{z^*} f$, we get for $T \in \partial'^{Cl}_{z^*} f(x)$ and k that

$$\tfrac{f(x+d)-f(x)}{\|d\|} \geq \tfrac{T(v_k)}{\|v_k\|} - \tfrac{\varepsilon}{k}.$$

By taking the limit for $k \to +\infty$, we get

$$T(d) \leq f(x + d) - f(x),\ \forall d \in \mathrm{dom}\, f - x$$

so, $T \in \partial_{\leq} f(x)$ and the equality follows since we always have that $\partial_{\leq} f(x) \subseteq \partial'^{Cl}_{z^*} f(x)$. $\square$

Corollary 26. *Let $(X, \|\cdot\|)$ be a Banach space and $(Z, \|\cdot\|)$ be a separable normed space ordered by a pointed closed convex cone Z_+ with nonempty interior, and let f be a lower semicontinuous function, $f : X \to Z \cup \{+\infty\}$. Then we have that $\partial'^{Cl}_{z^*} f$ is monotone for each $z^* \in Z_+^* \setminus \{0\}$ if and only if f is convex and $\partial_{\leq} f(x) = w\partial_{\not>} f(x)$ for each $x \in X$.*

Proof. From the precedent corollary we get that $\partial'^{Cl}_{z^*} f$ is monotone for each $z^* \in Z_+^* \setminus \{0\}$ if and only if for all $x \in X$

$$\bigcup_{z^* \in Z_+^* \setminus \{0\}} \partial'^{Cl}_{z^*} f(x) = \partial_{\leq} f(x).$$

Proposition 23 implies that $z^* \circ f$ is convex for each $z^* \in Z_+^* \setminus \{0\}$, thus f is convex. For a convex function we say that $\partial'^{Cl}_{z^*} f(x) = \partial_{z^*} f(x)$ and Proposition

14 gives that
$$w\partial_{\not>} f(x) = \bigcup_{z^*\in Z_+^*\setminus\{0\}} \partial_{z^*} f(x).$$

Thus, we derive $w\partial_{\not>} f(x) = \partial_{\leq} f(x)$.

Now, if f is convex, the equality $w\partial_{\not>} f(x) = \partial_{\leq} f(x)$ implies that
$$\bigcup_{z^*\in Z_+^*\setminus\{0\}} \partial'^{Cl}_{z^*} f(x) = \partial_{\leq} f(x)$$
and the precedent corollary leads to the conclusion. □

COROLLARY 27. *If, in addition, we suppose in the precedent corollary that $\partial_{\leq} f(x) \neq \emptyset$ for each $x \in \operatorname{dom} f$, then $\partial'^{Cl}_{z^*} f$ is monotone for each $z^* \in Z_+^* \setminus \{0\}$ if and only if f is convex and the order is total.*

The proof follows from the precedent corollary and from Proposition 15.

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