REFINEMENTS OF JENSEN-MERCER’S INEQUALITY FOR INDEX
SET FUNCTIONS WITH APPLICATIONS

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Abstract. Some refinements of Jensen-Mercer’s inequality are presented. They
are used to refine few inequalities among various means of Mercer’s type, and
they are further generalized for linear functionals.


Keywords. Jensen-Mercer’s inequality, index set function, means of Mercer’s
type.

1. INTRODUCTION

In paper [3] A. McD. Mercer proved the following variant of Jensen’s in-
equality, to which we will refer as to the “Jensen-Mercer’s inequality”.

THEOREM A. Let \( [a, b] \) be an interval in \( \mathbb{R} \), and \( x_1, \ldots, x_n \in [a, b] \). Let
\( w_1, \ldots, w_n \) be nonnegative real numbers such that \( W_n = \sum_{i=1}^n w_i > 0 \). If \( f \) is
a convex function on \( [a, b] \), then

\[
 f \left( a + b - \frac{1}{W_n} \sum_{i=1}^n w_i x_i \right) \leq f(a) + f(b) - \frac{1}{W_n} \sum_{i=1}^n w_i f(x_i). 
\]

In this paper we give some refinements of (1) and we present several ap-
lications of them. In Section 2 we first prove the Jensen-Mercer’s inequality
for weights satisfying conditions as for the reversed Jensen’s inequality (see
for example [4, p. 83]), and after that we prove refinements of Theorem A,
using an index set function. In Section 3 we use these results to refine some
well known inequalities among arithmetic, geometric, harmonic, power and
quasi-arithmetic means of Mercer’s type. In Section 4 we generalize our main
results for linear isotonic functionals.

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2. MAIN RESULTS

**Theorem 2.1.** Let \( w_1, \ldots, w_n \) be real numbers such that
\[
(2) \qquad w_1 > 0, \quad w_i \leq 0 \quad \text{for} \quad i = 2, \ldots, n, \quad W_n > 0.
\]
Let \([a, b]\) be an interval in \( \mathbb{R} \), and \( x_1, \ldots, x_n \in [a, b] \) such that \( \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \in [a, b] \). If \( f \) is a convex function on \([a, b]\), then (1) holds.

To prove Theorem 2.1, we need the following Lemma:

**Lemma 2.2.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function. Then for any \( x \in [a, b] \)
\[
f(a + b - x) \leq f(a) + f(b) - f(x).
\]

**Proof.** For every \( x \in [a, b] \), there exists a unique \( \lambda \in [0, 1] \) such that \( x = \lambda a + (1 - \lambda) b \). Since \( f \) is convex, we have
\[
f(a + b - x) = f(a + b - \lambda a - (1 - \lambda) b) = f((1 - \lambda) a + \lambda b) \leq (1 - \lambda) f(a) + \lambda f(b)
\]
\[
= f(a) + f(b) - (\lambda f(a) + (1 - \lambda) f(b)) \leq f(a) + f(b) - f(\lambda a + (1 - \lambda) b)
\]
\[
= f(a) + f(b) - f(x).
\]

**Proof of Theorem 2.1.** Weights \( w_1, \ldots, w_n \) satisfy conditions (2) and \( \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \in [a, b] \), so by Lemma 2.2 and by the reversed Jensen’s inequality, we have
\[
f\left(a + b - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right) \leq f(a) + f(b) - f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right)
\]
\[
\leq f(a) + f(b) - \frac{1}{W_n} \sum_{i=1}^{n} w_i f(x_i).
\]

Let \( I \) be a finite nonempty set of positive integers, and let \( f : [a, b] \to \mathbb{R} \). Let \( \mathbf{w} = \{w_i\}_{i \in I}, \mathbf{x} = \{x_i\}_{i \in I} \) be real sequences such that \( x_i \in [a, b] \) for all \( i \in I \), and \( A_I(\mathbf{x}, \mathbf{w}) = \frac{1}{W_I} \sum_{i \in I} w_i x_i \in [a, b] \), where \( W_I = \sum_{i \in I} w_i \). If we define the index set function \( F \) as
\[
F(I) = W_I \left[f(a) + f(b) - \frac{1}{W_I} \sum_{i \in I} w_i f(x_i) - f\left(a + b - \frac{1}{W_I} \sum_{i \in I} w_i x_i\right)\right],
\]
then the following theorem is valid.

**Theorem 2.3.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function. Let \( I \) and \( J \) be finite nonempty sets of positive integers such that \( I \cap J = \emptyset \). Let \( \mathbf{w} = \{w_i\}_{i \in I \cup J}, \mathbf{x} = \{x_i\}_{i \in I \cup J} \) be real sequences such that \( x_i \in [a, b] \) (\( i \in I \cup J \)), \( W_{I \cup J} > 0 \), and \( A_S(\mathbf{x}, \mathbf{w}) \in [a, b] \) (\( S = I, J, I \cup J \)). If \( W_I > 0 \) and \( W_J > 0 \), then
\[
(3) \quad F(I \cup J) \geq F(I) + F(J).
\]
If $W_I \cdot W_J < 0$, then the inequality (3) is reversed.

Proof. Since $f$ is convex, the same is also true for the function $g : [a, b] \to \mathbb{R}$ defined as $g(y) = f(a + b - y)$, $y \in [a, b]$. Hence, the following inequality holds for every $y_1, y_2 \in [a, b]$ and $u_1, u_2 > 0$

$$g \left( \frac{u_1 y_1 + u_2 y_2}{u_1 + u_2} \right) \leq \frac{u_1 g(y_1) + u_2 g(y_2)}{u_1 + u_2},$$

(4)

i.e.,

$$\left( u_1 + u_2 \right) f \left( a + b - \frac{u_1 y_1 + u_2 y_2}{u_1 + u_2} \right) \leq u_1 f (a + b - y_1) + u_2 f (a + b - y_2).$$

(5)

If $u_1 > 0$, $u_2 < 0$, $u_1 + u_2 > 0$ and $\frac{u_1 y_1 + u_2 y_2}{u_1 + u_2} \in [a, b]$, then (4), i.e., (5) is reversed. This is a simple consequence of (4) after we make the substitutions $u_1 \to u_1 + u_2$, $u_2 \to -u_2$, $y_1 \to \frac{u_1 y_1 + u_2 y_2}{u_1 + u_2}$, and $y_2 \to y_2$ (similarly as in the proof of the reversed Jensen’s inequality).

Suppose that $W_I > 0$ and $W_J > 0$. If we let

$$u_1 = W_I, \ u_2 = W_J, \ y_1 = A_f (x, w), \ y_2 = A_J (x, w)$$

in (5), then we obtain

$$W_{I \cup J} f (a + b - A_{I \cup J} (x, w)) \leq W_I f (a + b - A_f (x, w)) + W_J f (a + b - A_J (x, w)).$$

Multiplying the above inequality by $(-1)$ and adding to the both sides the term

$$W_{I \cup J} \left[ f(a) + f(b) - \frac{1}{W_{I \cup J}} \sum_{i \in I \cup J} w_i f(x_i) \right],$$

it follows that

$$W_{I \cup J} \left[ f(a) + f(b) - \frac{1}{W_{I \cup J}} \sum_{i \in I \cup J} w_i f(x_i) - f \left( a + b - \frac{1}{W_{I \cup J}} \sum_{i \in I \cup J} w_i x_i \right) \right] \geq W_I \left[ f(a) + f(b) - \frac{1}{W_I} \sum_{i \in I} w_i f(x_i) - f \left( a + b - \frac{1}{W_I} \sum_{i \in I} w_i x_i \right) \right]$$

$$+ W_J \left[ f(a) + f(b) - \frac{1}{W_J} \sum_{i \in J} w_i f(x_i) - f \left( a + b - \frac{1}{W_J} \sum_{i \in J} w_i x_i \right) \right].$$

In case when $W_I \cdot W_J < 0$, for instance $W_I > 0$ and $W_J < 0$, we again let

$$u_1 = W_I, \ u_2 = W_J, \ y_1 = A_f (x, w), \ y_2 = A_J (x, w),$$

and reversed (3) follows from reversed (5).

\[ \square \]

Corollary 2.4. Let $f : [a, b] \to \mathbb{R}$ be a convex function. Let $I_1, \ldots, I_k$ be finite nonempty sets of positive integers such that $I_i \cap I_j = \emptyset$, for all
\(i \neq j \in \{1, \ldots, k\}\). Let \(\mathbf{w} = \{w_i\}_{i \in \bigcup_{j=1}^k I_j}, \mathbf{x} = \{x_i\}_{i \in \bigcup_{j=1}^k I_j}\) be real sequences such that \(x_i \in [a, b]\) \((i \in \bigcup_{j=1}^k I_j)\), \(W_{\bigcup_{j=1}^k I_j} > 0\), and \(A_S (\mathbf{x}, \mathbf{w}) \in [a, b]\) 
\(S = I_1, \ldots, I_k, \bigcup_{j=1}^l I_j \ (l = 2, \ldots, n)\) If \(W_{I_j} > 0 \ (j = 1, \ldots, k)\), then

\[
F \left( \bigcup_{j=1}^k I_j \right) \geq \sum_{j=1}^k F (I_j).
\]

If \(W_{I_1} > 0\) and \(W_{I_j} < 0 \ (j = 2, \ldots, k)\), then the inequality \((6)\) is reversed.

**Proof.** Directly from Theorem 2.3 by induction. \(\square\)

The following corollaries give refinements of Theorem A.

**Corollary 2.5.** Let \(f : [a, b] \to \mathbb{R}\) be a convex function and \(I_k = \{1, \ldots, k\}\) 
\((k = 1, \ldots, n)\). Let \(\mathbf{w} = \{w_i\}_{i \in I_n}, \mathbf{x} = \{x_i\}_{i \in I_n}\) be real sequences such that
\(x_i \in [a, b]\) \((i \in I_n)\), and \(w_1 > 0\).

If \(w_i \geq 0 \ for i = 2, \ldots, n\), then

\[
F(I_n) \geq F(I_{n-1}) \geq \cdots \geq F(I_2) \geq F(I_1) \geq 0.
\]

If \(w_i \leq 0 \ for i = 2, \ldots, n\), \(W_{I_n} > 0\) and \(A_{I_n} (\mathbf{x}, \mathbf{w}) \in [a, b]\), then

\[
0 \leq F(I_n) \leq F(I_{n-1}) \leq \cdots \leq F(I_2) \leq F(I_1).
\]

**Proof.** Suppose that \(w_i \geq 0 \ for i = 2, \ldots, n\). First we show that

\[
F \{\{k\}\} = w_k [f (a) + f (b) - f (x_k)] \geq 0
\]

for any \(k \in I_n\). By Lemma 2.2 we have \(f (a + b - x_k) \leq f (a) + f (b) - f (x_k)\), and since \(w_k \geq 0\), it follows that \(F \{\{k\}\} \geq 0\). Now, by Theorem 2.3

\[
F(I_k) = F(I_{k-1} \cup \{k\}) \geq F(I_{k-1}) + F \{\{k\}\} \geq F(I_{k-1})
\]

for all \(k \in \{2, \ldots, n\}\).

Suppose that \(w_i \leq 0 \ for i = 2, \ldots, n\), \(W_{I_n} > 0\) and \(A_{I_n} (\mathbf{x}, \mathbf{w}) \in [a, b]\). First we show that from \(A_{I_n} (\mathbf{x}, \mathbf{w}) \in [a, b]\) it follows that \(A_{I_{n-1}} (\mathbf{x}, \mathbf{w}) \in [a, b]\). If we multiply inequality

\[
a \leq A_{I_n} (\mathbf{x}, \mathbf{w}) \leq b
\]

by \(W_{I_n} > 0\), and then add to the both sides \(-w_n x_n\), we obtain

\[
W_{I_n} a - w_n x_n \leq \sum_{i \in I_{n-1}} w_i x_i \leq W_{I_n} b - w_n x_n.
\]

Multiplying the above inequality by \(\frac{1}{W_{I_{n-1}}} > 0\), we have

\[
\frac{1}{W_{I_{n-1}}} (W_{I_{n-1}} a + w_n a - w_n x_n) \leq A_{I_{n-1}} (\mathbf{x}, \mathbf{w}) \leq \frac{1}{W_{I_{n-1}}} (W_{I_{n-1}} b + w_n b - w_n x_n).
\]
i.e.,
\[ a + \frac{w_n}{W_{i-1}} (a - x_n) \leq A_{I_{n-1}} (x, \mathbf{w}) \leq b + \frac{w_n}{W_{I_{n-1}}} (b - x_n). \]

Since, \( \frac{w_n}{W_{I_{n-1}}} (a - x_n) \geq 0 \) and \( \frac{w_n}{W_{I_{n-1}}} (b - x_n) \leq 0 \), it follows that
\[ a \leq A_{I_{n-1}} (x, \mathbf{w}) \leq b. \]

By iteration we obtain \( A_{I_k} (x, \mathbf{w}) \in [a, b] \) for all \( k \in \{2, \ldots, n-1\} \). Similarly as before we have \( F(\{k\}) \leq 0 \) for any \( k \in \{2, \ldots, n\} \). Now, by Theorem 2.3
\[ F(I_k) = F(I_{k-1} \cup \{k\}) \leq F(I_{k-1}) + F(\{k\}) \leq F(I_{k-1}) \]
for all \( k \in \{2, \ldots, n\} \), and finally, by Theorem 2.1 \( F(I_n) \geq 0 \). \( \square \)

**Corollary 2.6.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function and \( I_k = \{1, \ldots, k\} \) \( (k = 1, \ldots, n) \). Let \( \mathbf{w} = \{w_i\}_{i \in I_n}, \mathbf{x} = \{x_i\}_{i \in I_n} \) be real sequences such that \( x_i \in [a,b] \) \( (i \in I_n) \).

If \( w_i > 0 \) for all \( i = 1, \ldots, n \), then
\[ F(I_n) \geq \max_{1 \leq i \leq j \leq n} \left\{ (w_i + w_j) \left[ f(a) + f(b) - \frac{w_if(x_i) + w_jf(x_j)}{w_i + w_j} \right] - f \left( a + b - \frac{w_ix_i + w_jx_j}{w_i + w_j} \right) \right\}, \tag{8} \]
and
\[ F(I_n) \geq \max_{1 \leq i \leq n} \{w_i [f(a) + f(b) - f(x_i) - f (a + b - x_i)]\}. \tag{9} \]

If \( \mathbf{w} \) satisfy \( 2 \) and \( A_{I_n} (x, \mathbf{w}) \in [a, b] \), then
\[ F(I_n) \leq \min_{2 \leq j \leq n} \left\{ (w_1 + w_j) \left[ f(a) + f(b) - \frac{w_1f(x_1) + w_jf(x_j)}{w_1 + w_j} \right] - f \left( a + b - \frac{w_1x_1 + w_jx_j}{w_1 + w_j} \right) \right\}, \tag{10} \]

**Proof.** Suppose that \( w_i > 0 \) for all \( i = 1, \ldots, n \). Similarly as \( F(I_n) \geq F(I_2) \) in Corollary 2.3 we may conclude that
\[ F(I_n) \geq F(\{i, j\}) \] for all \( i \neq j \in \{1, \ldots, n\} \),
so the inequality \( 8 \) immediately follows. From \( 11 \) we have \( F(I_n) \geq F(\{i\}) \) for all \( i \in \{1, \ldots, n\} \), so the inequality \( 9 \) is also proved.

The inequality \( 10 \) can be proved in the similar way. \( \square \)

**Remark 2.7.** Analogous assertions can be formulated for concave functions using the fact that \( f \) is concave iff \( -f \) is convex. \( \square \)
3. APPLICATIONS

Let $A_n$, $G_n$, $H_n$, and $M^{[r]}_n$ be the arithmetic, geometric, harmonic, and power mean of order $r$, respectively, of the real numbers $x_i \in [a,b]$, where $0 < a < b$, formed with the positive weights $w_i$ ($i = 1, \ldots, n$). For the various properties of these means and relations among them we refer the reader to [2]. For example, it is well known that

$$\left( \frac{A_n}{G_n} \right)^{W_n} \geq \left( \frac{A_{n-1}}{G_{n-1}} \right)^{W_{n-1}} \geq \cdots \geq \left( \frac{A_1}{G_1} \right)^{W_1} \geq 1,$$

$$W_n \left( A_n - G_n \right) \geq W_{n-1} \left( A_{n-1} - G_{n-1} \right) \geq \cdots \geq W_1 \left( A_1 - G_1 \right) \geq 0.$$

If we define

$$\tilde{A}_n := a + b - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i = a + b - A_n,$$

$$\tilde{G}_n := \frac{ab}{\left( \prod_{i=1}^{n} x_i^{w_i} \right)^{\frac{1}{W_n}}},$$

$$\tilde{H}_n := \left( a^{-1} + b^{-1} - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i^{-1} \right)^{-1} = \left( a^{-1} + b^{-1} - H_n^{-1} \right)^{-1},$$

$$\tilde{M}^{[r]}_n := \left( a^r + b^r - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i^r \right)^{\frac{1}{r}} = \left( a^r + b^r - \left( M^{[r]}_n \right)^r \right)^{\frac{1}{r}},$$

then we have the following results.

**Theorem 3.1.**

(i)  

\[(12) \quad \left( \frac{\tilde{A}_n}{\tilde{G}_n} \right)^{W_n} \geq \left( \frac{\tilde{A}_{n-1}}{\tilde{G}_{n-1}} \right)^{W_{n-1}} \geq \cdots \geq \left( \frac{\tilde{A}_1}{\tilde{G}_1} \right)^{W_1} \geq 1, \]

(ii)  

$$W_n \left( \tilde{A}_n - \tilde{G}_n \right) \geq W_{n-1} \left( \tilde{A}_{n-1} - \tilde{G}_{n-1} \right) \geq \cdots \geq W_1 \left( \tilde{A}_1 - \tilde{G}_1 \right) \geq 0.$$  

**Proof.** (i) Applying Corollary 2.5 to the convex function $f(x) = -\ln x$, we obtain

$$\ln \left( \frac{\tilde{A}_n}{\tilde{G}_n} \right)^{W_n} \geq \ln \left( \frac{\tilde{A}_{n-1}}{\tilde{G}_{n-1}} \right)^{W_{n-1}} \geq \cdots \geq \ln \left( \frac{\tilde{A}_1}{\tilde{G}_1} \right)^{W_1} \geq 0,$$

from which (12) follows.

(ii) Applying Corollary 2.5 to the convex function $f(x) = \exp x$, and replacing $a$, $b$, and $x_i$ with $\ln a$, $\ln b$, and $\ln x_i$ respectively, we obtain

$$W_n \left( \tilde{A}_n - \tilde{G}_n \right) \geq W_{n-1} \left( \tilde{A}_{n-1} - \tilde{G}_{n-1} \right) \geq \cdots \geq W_1 \left( \tilde{A}_1 - \tilde{G}_1 \right) \geq 0,$$
since in this case
\[ F(I_k) = W_k \left( a + b - \frac{1}{W_k} \sum_{i=1}^{k} w_i x_i - \exp \left( \ln a + \ln b - \frac{1}{W_k} \sum_{i=1}^{k} w_i \ln x_i \right) \right) \]
\[ = W_k \left( \tilde{A}_k - \tilde{G}_k \right). \]

**Corollary 3.2.**
(i) \[ \left( \frac{G_n}{H_n} \right)^{W_n} \geq \left( \frac{G_{n-1}}{H_{n-1}} \right)^{W_{n-1}} \geq \cdots \geq \left( \frac{G_1}{H_1} \right)^{W_1} \geq 1, \]
(ii) \[ W_n \left( \frac{1}{H_n} - \frac{1}{G_n} \right) \geq W_n \left( \frac{1}{H_{n-1}} - \frac{1}{G_{n-1}} \right) \geq \cdots \geq W_1 \left( \frac{1}{H_1} - \frac{1}{G_1} \right) \geq 0. \]

**Proof.** Directly from Theorem 3.1 by the substitutions \( a \to \frac{1}{a}, b \to \frac{1}{b}, \)
\( x_i \to \frac{1}{x_i}. \)

**Theorem 3.3.** For \( r \leq 1, \)
\[ W_n \left( \tilde{A}_n - \tilde{M}_k^r \right) \geq W_{n-1} \left( \tilde{A}_{n-1} - \tilde{M}_{n-1}^r \right) \geq \cdots \geq W_1 \left( \tilde{A}_1 - \tilde{M}_1^r \right) \geq 0. \]

For \( r \geq 1, \) the inequalities \[ (13) \]
are reversed.

**Proof.** Suppose that \( r \leq 1. \) Applying Corollary 2.3 to the convex function \( f(x) = x^\frac{r}{2}, \)
and replacing \( a, b, \) and \( x_i \) with \( a^r, b^r, \) and \( x_i^r \) respectively, we obtain \[ (13) \] since in this case
\[ F(I_k) = W_k \left( a + b - \frac{1}{W_k} \sum_{i=1}^{k} w_i x_i - \left( a^r + b^r - \frac{1}{W_k} \sum_{i=1}^{k} w_i x_i^r \right) \right) \]
\[ = W_k \left( \tilde{A}_k - \tilde{M}_k^r \right). \]

If \( r \geq 1, \) then the function \( f(x) = x^\frac{1}{r} \) is concave, so the inequalities \[ (13) \] are reversed.

**Corollary 3.4.**
\[ W_n \left( \tilde{A}_n - \tilde{H}_n \right) \geq W_{n-1} \left( \tilde{A}_{n-1} - \tilde{H}_{n-1} \right) \geq \cdots \geq W_1 \left( \tilde{A}_1 - \tilde{H}_1 \right) \geq 0. \]

**Remark 3.5.** Obviously, the assertion (ii) from Theorem 3.1 is also direct consequence of Theorem 3.3.

**Theorem 3.6.** Let \( r, s \in \mathbb{R}, r \leq s. \)

(i) If \( s > 0, \) then
\[ W_n \left( \left( \tilde{M}_n^s \right)^{\frac{s}{r}} - \left( \tilde{M}_k^r \right)^{\frac{s}{r}} \right) \geq W_{n-1} \left( \left( \tilde{M}_{n-1}^s \right)^{\frac{s}{r}} - \left( \tilde{M}_{n-1}^r \right)^{\frac{s}{r}} \right) \]
\[ \geq \cdots \geq W_1 \left( \left( \tilde{M}_1^s \right)^{\frac{s}{r}} - \left( \tilde{M}_1^r \right)^{\frac{s}{r}} \right) \geq 0, \]
(ii) If \( s < 0 \), then the inequalities (14) are reversed.

**Proof.** Suppose that \( s > 0 \). Applying Corollary 2.5 to the convex function \( f(x) = x^\frac{s}{n} \), and replacing \( a, b, \) and \( x_i \) with \( a^r, b^s, \) and \( x_i^s \) respectively, we obtain (14) since

\[
F(I_k) = W_k \left( a^s + b^s - \frac{1}{W_n} \sum_{i=1}^{k} w_i x_i^s - \left( a^r + b^s - \frac{1}{W_n} \sum_{i=1}^{k} w_i x_i^r \right) \right)
= W_k \left( (\tilde{M}_k^{[s]})^s - (\tilde{M}_k^{[r]})^s \right).
\]

If \( s < 0 \), then the function \( f(x) = x^\frac{s}{n} \) is concave, so (14) is reversed. \( \Box \)

Let \( \varphi : [a, b] \to \mathbb{R} \) be a strictly monotonic and continuous function, where \([a, b] \subset \mathbb{R} \). Then for a given \( n \)-tuple \( x = (x_1, \ldots, x_n) \in [a, b]^n \) and positive \( n \)-tuple \( w = (w_1, \ldots, w_n) \in \mathbb{R}^n \), the value

\[
M^{[n]}_{\varphi} = \varphi^{-1} \left( \frac{1}{W_n} \sum_{i=1}^{n} w_i \varphi(x_i) \right)
\]

is well defined and is called quasi-arithmetic mean of \( x \) with weights \( w \) (see for example \([2\) p. 215]). If we define

\[
\tilde{M}^{[n]}_{\varphi} := \varphi^{-1} \left( \varphi(a) + \varphi(b) - \frac{1}{W_n} \sum_{i=1}^{n} w_i \varphi(x_i) \right),
\]

then we have the following results.

**Theorem 3.7.** Let \( \varphi, \psi : [a, b] \to \mathbb{R} \) be strictly monotonic and continuous functions. If \( \psi \circ \varphi^{-1} \) is convex on \([a, b] \), then

\[
W_n \left( \psi \left( \tilde{M}^{[n]}_{\psi} \right) - \psi \left( \tilde{M}^{[n]}_{\varphi} \right) \right) \geq W_{n-1} \left( \psi \left( \tilde{M}^{[n-1]}_{\psi} \right) - \psi \left( \tilde{M}^{[n-1]}_{\varphi} \right) \right) \geq \cdots \geq W_1 \left( \psi \left( \tilde{M}^{[1]}_{\psi} \right) - \psi \left( \tilde{M}^{[1]}_{\varphi} \right) \right) \geq 0.
\]

If \( \psi \circ \varphi^{-1} \) is concave on \([a, b] \), then the inequalities (15) are reversed.

**Proof.** Applying Corollary 2.5 to the convex function \( f = \psi \circ \varphi^{-1} \), and replacing \( a, b, \) and \( x_i \) with \( \varphi(a), \varphi(b), \) and \( \varphi(x_i) \) respectively, we obtain (15) since in this case

\[
F(I_k) = W_k \left( \varphi(a) + \varphi(b) - \frac{1}{W_k} \sum_{i=1}^{k} w_i \varphi(x_i) \right)
- \left( \varphi \circ \varphi^{-1} \right) \left( \varphi(a) + \varphi(b) - \frac{1}{W_k} \sum_{i=1}^{k} w_i \varphi(x_i) \right) \right)
= W_k \left( \psi \left( \tilde{M}^{[k]}_{\psi} \right) - \psi \left( \tilde{M}^{[k]}_{\varphi} \right) \right). \]

**Remark 3.8.** Theorems 3.1, 3.3 and 3.6 follow from Theorem 3.7 by choosing adequate functions \( \varphi \) and \( \psi \), and appropriate substitutions.
Corollary 3.9. Let $\varphi, \psi : [a, b] \to \mathbb{R}$ be strictly monotonic and continuous functions. If $\psi \circ \varphi^{-1}$ is convex on $[a, b]$, then
\[
W_n \left( \psi \left( \overline{M}^{[n]} \right) - \psi \left( \overline{M}^{[n]} \right) \right) \\
\geq \max_{1 \leq i < j \leq n} \left\{ (w_i + w_j) \left[ \psi(a) + \psi(b) - \frac{w_j \psi(x_i) + w_i \psi(x_j)}{w_i + w_j} \right. \right. \\
\left. \left. - \left( \psi \circ \varphi^{-1} \right) \left( \varphi(a) + \varphi(b) - \frac{w_j \varphi(x_i) + w_i \varphi(x_j)}{w_i + w_j} \right) \right] \right\},
\]
and
\[
W_n \left( \psi \left( \overline{M}^{[n]} \right) - \psi \left( \overline{M}^{[n]} \right) \right) \\
\geq \max_{1 \leq i \leq n} \left\{ w_i \left[ \psi(a) + \psi(b) - \psi(x_i) - \left( \psi \circ \varphi^{-1} \right) \left( \varphi(a) + \varphi(b) - \varphi(x_i) \right) \right] \right\}.
\]
If $\psi \circ \varphi^{-1}$ is concave on $[a, b]$, then the inequalities (16) and (17), with max replaced by min, are reversed.

Remark 3.10. Analogous assertions can be formulated for the means of Mercer’s type formed with weights satisfying (2). \qed

4. Further Generalization

Let $E$ be a nonempty set, $\mathcal{A}$ be an algebra of subsets of $E$, and $L$ be a linear class of real valued functions $f : E \to \mathbb{R}$ having the properties:

L1: $f, g \in L \Rightarrow (\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
L2: $1 \in L$, i.e., if $f(t) = 1$ for $t \in E$, then $f \in L$;
L3: $f \in L, E_1 \in \mathcal{A} \Rightarrow f \cdot \chi_{E_1} \in L$,

where $\chi_{E_1}$ is the indicator function of $E_1$. It follows from L2, L3 that $\chi_{E_1} \in L$ for every $E_1 \in \mathcal{A}$.

Let $A : L \to \mathbb{R}$ be an isotonic linear functional having the properties:

A1: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L, \alpha, \beta \in \mathbb{R}$;
A2: $f \in L, f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$;
A3: $A(1) = 1$.

It follows from L3 that for every $E_1 \in \mathcal{A}$ such that $A(\chi_{E_1}) > 0$, the functional $A_1$ defined for all $f \in L$ as $A(f \chi_{E_1}) / A(\chi_{E_1})$ is an isotonic linear functional with $A_1(1) = 1$. Furthermore, we observe that
\[
A(\chi_{E_1}) + A\left(\chi_{E_1 \setminus E_1}\right) = 1,
\]
and
\[
A(f) = A(f \cdot \chi_{E_1}) + A(f \cdot \chi_{E_1 \setminus E_1}).
\]

Let $\varphi : [a, b] \to \mathbb{R}$ be a continuous function, where $[a, b] \subset \mathbb{R}$. In [4], under the above assumptions, the following variant of the Jessen’s inequality
Hence, we have
\[(20) \quad \varphi(a + b - A(f)) \leq \varphi(a) + \varphi(b) - A(\varphi(f));\]
if \(\varphi\) is concave, then the inequality \((20)\) is reversed.

For \(E_1 \in A\) and \(f \in L\) such that \(\varphi(f) \in L\), we define the function \(F_f : A \to \mathbb{R}\) with
\[
F_f(E_1) = A(\chi_{E_1}) \left[ \varphi(a) + \varphi(b) - \frac{A(\varphi(f) \cdot \chi_{E_1})}{A(\chi_{E_1})} - \varphi \left( a + b - \frac{A(f \cdot \chi_{E_1})}{A(\chi_{E_1})} \right) \right].
\]

**Theorem 4.1.** Under the above assumptions, if \(\varphi\) is convex, then
\[(21) \quad F_f(E) \geq F_f(E_1) + F_f(E \setminus E_1) \geq F_f(E_1) \geq 0 \]
for all \(E_1 \in A\) such that \(0 < A(\chi_{E_1}) < 1\).

**Proof.** Since \(\varphi\) is continuous and convex, the same is also true for the function
\[\psi : [a, b] \to \mathbb{R}\]
defined as
\[\psi(t) = \varphi(a + b - t), \quad t \in [a, b].\]
Hence, the following inequality holds for every \(t_1, t_2 \in [a, b]\) and \(p, q \in \mathbb{R}\) such that \(p + q = 1\)
\[p\psi(t_1) + q\psi(t_2) \geq \psi(pt_1 + qt_2),\]
i.e.,
\[p\varphi(a + b - t_1) + q\varphi(a + b - t_2) \geq \varphi(a + b - (pt_1 + qt_2)).\]
If we let \(p = A(\chi_{E_1}), q = A(\chi_{E \setminus E_1}), t_1 = \frac{A(f \cdot \chi_{E_1})}{A(\chi_{E_1})},\) and \(t_2 = \frac{A(f \cdot \chi_{E \setminus E_1})}{A(\chi_{E \setminus E_1})}\), then
\[p + q = 1\] by the equality \((18)\), and \(pt_1 + qt_2 = A(f)\) by the equality \((19)\).
Similarly, we can use that
\[A(\varphi(f)) = A(\varphi(f) \cdot \chi_{E_1}) + A(\varphi(f) \cdot \chi_{E \setminus E_1}).\]
Hence, we have
\[A(\chi_{E_1}) \varphi \left( a + b - \frac{A(f \cdot \chi_{E_1})}{A(\chi_{E_1})} \right) + A(\chi_{E \setminus E_1}) \varphi \left( a + b - \frac{A(f \cdot \chi_{E \setminus E_1})}{A(\chi_{E \setminus E_1})} \right) \geq \varphi(a + b - A(f)).\]
Multiplying the above inequality with \((-1)\) and adding to the both sides the term
\[\varphi(a) + \varphi(b) - A(\varphi(f))\]
we obtain the first inequality in \((21)\).

To prove the remaining two inequalities in \((21)\), observe that they are simple consequences of \((20)\) applied to the isotonic linear functional \(A_1\) defined for \(f \in L\) as \(\frac{A(f \cdot \chi_{E_1})}{A(\chi_{E_1})}\) and having \(A_1(1) = 1\). \(\Box\)
Corollary 4.2. Let $\{E_1, \ldots, E_n\}$ be a partition of $E$ (i.e., $E = \bigcup_{j=1}^n E_j, E_i \cap E_j = \emptyset$ for all $i \neq j \in \{1, \ldots, n\}$) such that $0 < A(\chi_{E_i}) < 1$ for all $j \in \{1, \ldots, n\}$. If $\varphi$ is convex, then

\begin{equation}
F_f(E) \geq \sum_{j=1}^n F_f(E_j),
\end{equation}

and

\begin{equation}
F_f(E) \geq F_f\left(\bigcup_{j=1}^{n-1} E_j\right) \geq F_f\left(\bigcup_{j=1}^{n-2} E_j\right) \\
\geq \cdots \geq F_f(E_1 \cup E_2) \geq F_f(E_1) \geq 0.
\end{equation}

Proof. Directly from Theorem 4.1 by induction. \qed

Remark 4.3. If $\varphi$ is concave, then the inequalities (21)–(23) are reversed. \qed

Corollary 4.4. Let $\{E_1, \ldots, E_n\}$ be a partition of $E$ such that $0 < A(\chi_{E_i}) < 1$ for all $j \in \{1, \ldots, n\}$. If $\varphi$ is convex, then

\begin{equation}
F_f(E) \geq \max_{1 \leq i < j \leq n} \{F_f(E_i \cup E_j)\},
\end{equation}

and

\begin{equation}
F_f(E) \geq \max_{1 \leq j \leq n} \{F_f(E_j)\}.
\end{equation}

If $\varphi$ is concave, then the inequalities (24) and (25), with max replaced by min, are reversed.

Remark 4.5. We also may obtain similar results as in Theorem 3.7 and Corollary 3.9, for the generalized quasi-arithmetic means of Mercer’s type defined in [1], as

$$\tilde{M}_\varphi(f, A) = \varphi^{-1}(\varphi(a) + \varphi(b) - A(\varphi(f))).$$

REFERENCES


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