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REFINEMENTS OF JENSEN-MERCER'S INEQUALITY FOR INDEX SET FUNCTIONS WITH APPLICATIONS

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Abstract. Some refinements of Jensen-Mercer's inequality are presented. They are used to refine few inequalities among various means of Mercer's type, and they are further generalized for linear functionals.

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1. INTRODUCTION

In paper [3] A. McD. Mercer proved the following variant of Jensen's inequality, to which we will refer as to the "Jensen-Mercer's inequality".

THEOREM A. *Let* $[a,b]$ *be an interval in* $\mathbb{R}$*, and* $x_1,\dots,x_n \in [a,b]$*. Let* $w_1,\dots,w_n$ *be nonnegative real numbers such that* $W_n=\sum_{i=1}^n w_i>0$*. If* f *is a convex function on* $[a,b]$*, then*

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right)\leq f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^n w_i f\left(x_i\right). \tag{1}$$

In this paper we give some refinements of (1) and we present several applications of them. In Section 2 we first prove the Jensen-Mercer's inequality for weights satisfying conditions as for the reversed Jensen's inequality (see for example [4, p. 83]), and after that we prove refinements of Theorem A, using an index set function. In Section 3 we use these results to refine some well known inequalities among arithmetic, geometric, harmonic, power and quasi-arithmetic means of Mercer's type. In Section 4 we generalize our main results for linear isotonic functionals.

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2. MAIN RESULTS

THEOREM 2.1. *Let* $w_1, \ldots, w_n$ *be real numbers such that*

$$w_1 > 0, \quad w_i \leq 0 \text{ for } i = 2, \ldots, n, \quad W_n > 0. \tag{2}$$

Let $[a,b]$ *be an interval in* $\mathbb{R}$, *and* $x_1, \ldots, x_n \in [a,b]$ *such that* $\frac{1}{W_n}\sum_{i=1}^n w_i x_i \in [a,b]$. *If* f *is a convex function on* $[a,b]$, *then* (1) *holds.*

To prove Theorem 2.1, we need the following Lemma:

LEMMA 2.2. *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function. Then for any* $x \in [a,b]$

$$f(a+b-x) \leq f(a) + f(b) - f(x).$$

Proof. For every $x \in [a,b]$, there exists a unique $\lambda \in [0,1]$ such that $x = \lambda a + (1-\lambda)b$. Since f is convex, we have

$$\begin{aligned} f(a+b-x) &= f(a+b-\lambda a-(1-\lambda)b) \\ &= f((1-\lambda)a+\lambda b) \\ &\leq (1-\lambda)f(a)+\lambda f(b) \\ &= f(a)+f(b)-(\lambda f(a)+(1-\lambda)f(b)) \\ &\leq f(a)+f(b)-f(\lambda a+(1-\lambda)b) \\ &= f(a)+f(b)-f(x). \end{aligned}$$ □

Proof of Theorem 2.1. Weights $w_1, \ldots, w_n$ satisfy conditions (2) and $\frac{1}{W_n}\sum_{i=1}^n w_i x_i \in [a,b]$, so by Lemma 2.2 and by the reversed Jensen's inequality, we have

$$\begin{aligned} f\left(a+b-\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) &\leq f(a)+f(b)-f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \\ &\leq f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^n w_i f(x_i). \end{aligned}$$ □

Let I be a finite nonempty set of positive integers, and let $f : [a,b] \to \mathbb{R}$. Let $\mathbf{w} = \{w_i\}_{i\in I}$, $\mathbf{x} = \{x_i\}_{i\in I}$ be real sequences such that $x_i \in [a,b]$ for all $i \in I$, and $A_I(\mathbf{x},\mathbf{w}) = \frac{1}{W_I}\sum_{i\in I} w_i x_i \in [a,b]$, where $W_I = \sum_{i\in I} w_i$. If we define the index set function F as

$$F(I) = W_I\left[f(a)+f(b)-\frac{1}{W_I}\sum_{i\in I} w_i f(x_i) - f\left(a+b-\frac{1}{W_I}\sum_{i\in I} w_i x_i\right)\right],$$

then the following theorem is valid.

THEOREM 2.3. *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function. Let* I *and* J *be finite nonempty sets of positive integers such that* $I \cap J = \varnothing$. *Let* $\mathbf{w} = \{w_i\}_{i\in I\cup J}$, $\mathbf{x} = \{x_i\}_{i\in I\cup J}$ *be real sequences such that* $x_i \in [a,b]$ $(i \in I \cup J)$, $W_{I\cup J} > 0$, *and* $A_S(\mathbf{x},\mathbf{w}) \in [a,b]$ $(S = I, J, I\cup J)$. *If* $W_I > 0$ *and* $W_J > 0$, *then*

$$F(I\cup J) \geq F(I) + F(J). \tag{3}$$

If $W_I \cdot W_J < 0$, *then the inequality* (3) *is reversed.*

Proof. Since f is convex, the same is also true for the function $g : [a, b] \to \mathbb{R}$ defined as $g(y) = f(a+b-y)$, $y \in [a, b]$. Hence, the following inequality holds for every $y_1, y_2 \in [a, b]$ and $u_1, u_2 > 0$

$$g\left(\frac{u_1 y_1 + u_2 y_2}{u_1 + u_2}\right) \leq \frac{u_1 g(y_1) + u_2 g(y_2)}{u_1 + u_2}, \tag{4}$$

i.e.,

$$(u_1 + u_2) f\left(a + b - \frac{u_1 y_1 + u_2 y_2}{u_1 + u_2}\right) \leq u_1 f(a + b - y_1) + u_2 f(a + b - y_2). \tag{5}$$

If $u_1 > 0$, $u_2 < 0$, $u_1 + u_2 > 0$ and $\frac{u_1 y_1 + u_2 y_2}{u_1 + u_2} \in [a, b]$, then (4), i.e., (5) is reversed. This is a simple consequence of (4) after we make the substitutions $u_1 \to u_1 + u_2$, $u_2 \to -u_2$, $y_1 \to \frac{u_1 y_1 + u_2 y_2}{u_1 + u_2}$, and $y_2 \to y_2$ (similarly as in the proof of the reversed Jensen's inequality).

Suppose that $W_I > 0$ and $W_J > 0$. If we let

$$u_1 = W_I,\ u_2 = W_J,\ y_1 = A_I(\mathbf{x}, \mathbf{w}),\ y_2 = A_J(\mathbf{x}, \mathbf{w})$$

in (5), then we obtain

$$\begin{aligned} &W_{I\cup J} f(a + b - A_{I\cup J}(\mathbf{x}, \mathbf{w})) \\ &\leq W_I f(a + b - A_I(\mathbf{x}, \mathbf{w})) + W_J f(a + b - A_J(\mathbf{x}, \mathbf{w})). \end{aligned}$$

Multiplying the above inequality by (-1) and adding to the both sides the term

$$W_{I\cup J}\left[f(a) + f(b) - \frac{1}{W_{I\cup J}} \sum_{i\in I\cup J} w_i f(x_i)\right],$$

it follows that

$$\begin{aligned} &W_{I\cup J}\left[f(a) + f(b) - \frac{1}{W_{I\cup J}} \sum_{i\in I\cup J} w_i f(x_i) - f\left(a + b - \frac{1}{W_{I\cup J}} \sum_{i\in I\cup J} w_i x_i\right)\right] \\ &\geq W_I\left[f(a) + f(b) - \frac{1}{W_I} \sum_{i\in I} w_i f(x_i) - f\left(a + b - \frac{1}{W_I} \sum_{i\in I} w_i x_i\right)\right] \\ &\quad + W_J\left[f(a) + f(b) - \frac{1}{W_J} \sum_{i\in J} w_i f(x_i) - f\left(a + b - \frac{1}{W_J} \sum_{i\in J} w_i x_i\right)\right]. \end{aligned}$$

In case when $W_I \cdot W_J < 0$, for instance $W_I > 0$ and $W_J < 0$, we again let

$$u_1 = W_I,\ u_2 = W_J,\ y_1 = A_I(\mathbf{x}, \mathbf{w}),\ y_2 = A_J(\mathbf{x}, \mathbf{w}),$$

and reversed (3) follows from reversed (5). □

COROLLARY 2.4. *Let* $f : [a, b] \to \mathbb{R}$ *be a convex function. Let* $I_1, \ldots, I_k$ *be finite nonempty sets of positive integers such that* $I_i \cap I_j = \varnothing$, *for all*

$i \neq j \in \{1, \ldots, k\}$. Let $\mathbf{w} = \{w_i\}_{i \in \cup_{j=1}^k I_j}$, $\mathbf{x} = \{x_i\}_{i \in \cup_{j=1}^k I_j}$ be real sequences such that $x_i \in [a, b]$ $\left(i \in \cup_{j=1}^k I_j\right)$, $W_{\cup_{j=1}^k I_j} > 0$, and $A_S(\mathbf{x}, \mathbf{w}) \in [a, b]$ $\left(S = I_1, \ldots, I_k, \cup_{j=1}^l I_j \ \ (l = 2, \ldots, n)\right)$. If $W_{I_j} > 0$ $(j = 1, \ldots, k)$, then

$$F\left(\bigcup_{j=1}^{k} I_j\right) \geq \sum_{j=1}^{k} F(I_j). \tag{6}$$

If $W_{I_1} > 0$ and $W_{I_j} < 0$ $(j = 2, \ldots, k)$, then the inequality (6) *is reversed.*

Proof. Directly from Theorem 2.3 by induction. □

The following corollaries give refinements of Theorem A.

Corollary 2.5. *Let $f : [a, b] \to \mathbb{R}$ be a convex function and $I_k = \{1, \ldots, k\}$ $(k = 1, \ldots, n)$. Let $\mathbf{w} = \{w_i\}_{i \in I_n}$, $\mathbf{x} = \{x_i\}_{i \in I_n}$ be real sequences such that $x_i \in [a, b]$ $(i \in I_n)$, and $w_1 > 0$.*
If $w_i \geq 0$ for $i = 2, \ldots, n$, then

$$F(I_n) \geq F(I_{n-1}) \geq \cdots \geq F(I_2) \geq F(I_1) \geq 0. \tag{7}$$

If $w_i \leq 0$ for $i = 2, \ldots, n$, $W_{I_n} > 0$ and $A_{I_n}(\mathbf{x}, \mathbf{w}) \in [a, b]$, then

$$0 \leq F(I_n) \leq F(I_{n-1}) \leq \cdots \leq F(I_2) \leq F(I_1).$$

Proof. Suppose that $w_i \geq 0$ for $i = 2, \ldots, n$. First we show that

$$F(\{k\}) = w_k [f(a) + f(b) - f(x_k) - f(a + b - x_k)] \geq 0$$

for any $k \in I_n$. By Lemma 2.2, we have $f(a + b - x_k) \leq f(a) + f(b) - f(x_k)$, and since $w_k \geq 0$, it follows that $F(\{k\}) \geq 0$. Now, by Theorem 2.3,

$$F(I_k) = F(I_{k-1} \cup \{k\}) \geq F(I_{k-1}) + F(\{k\}) \geq F(I_{k-1})$$

for all $k \in \{2, \ldots, n\}$.

Suppose that $w_i \leq 0$ for $i = 2, \ldots, n$, $W_{I_n} > 0$ and $A_{I_n}(\mathbf{x}, \mathbf{w}) \in [a, b]$. First we show that from $A_{I_n}(\mathbf{x}, \mathbf{w}) \in [a, b]$ it follows that $A_{I_{n-1}}(\mathbf{x}, \mathbf{w}) \in [a, b]$. If we multiply inequality

$$a \leq A_{I_n}(\mathbf{x}, \mathbf{w}) \leq b$$

by $W_{I_n} > 0$, and then add to the both sides $-w_n x_n$, we obtain

$$W_{I_n} a - w_n x_n \leq \sum_{i \in I_{n-1}} w_i x_i \leq W_{I_n} b - w_n x_n.$$

Multiplying the above inequality by $\frac{1}{W_{I_{n-1}}} > 0$, we have

$$\frac{1}{W_{I_{n-1}}} \left(W_{I_{n-1}} a + w_n a - w_n x_n\right) \leq A_{I_{n-1}}(\mathbf{x}, \mathbf{w}) \leq \frac{1}{W_{I_{n-1}}} \left(W_{I_{n-1}} b + w_n b - w_n x_n\right),$$

i.e.,

$$a+\frac{w_n}{W_{I_{n-1}}}\left(a-x_n\right)\leq A_{I_{n-1}}\left(\mathbf{x},\mathbf{w}\right)\leq b+\frac{w_n}{W_{I_{n-1}}}\left(b-x_n\right).$$

Since, $\frac{w_n}{W_{I_{n-1}}}\left(a-x_n\right)\geq 0$ and $\frac{w_n}{W_{I_{n-1}}}\left(b-x_n\right)\leq 0$, it follows that

$$a\leq A_{I_{n-1}}\left(\mathbf{x},\mathbf{w}\right)\leq b.$$

By iteration we obtain $A_{I_k}\left(\mathbf{x},\mathbf{w}\right)\in[a,b]$ for all $k\in\{2,\ldots,n-1\}$. Similarly as before we have $F\left(\{k\}\right)\leq 0$ for any $k\in\{2,\ldots,n\}$. Now, by Theorem 2.3,

$$F(I_k)=F(I_{k-1}\cup\{k\})\leq F(I_{k-1})+F\left(\{k\}\right)\leq F(I_{k-1})$$

for all $k\in\{2,\ldots,n\}$, and finally, by Theorem 2.1, $F(I_n)\geq 0$. □

Corollary 2.6. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $I_k=\{1,\ldots,k\}$ $(k=1,\ldots,n)$. Let $\mathbf{w}=\{w_i\}_{i\in I_n}$, $\mathbf{x}=\{x_i\}_{i\in I_n}$ be real sequences such that $x_i\in[a,b]$ $(i\in I_n)$.*
If $w_i>0$ for all $i=1,\ldots,n$, then

$$F(I_n)\geq\max_{1\leq i<j\leq n}\left\{(w_i+w_j)\left[f(a)+f(b)-\frac{w_if\left(x_i\right)+w_jf\left(x_j\right)}{w_i+w_j}-f\left(a+b-\frac{w_ix_i+w_jx_j}{w_i+w_j}\right)\right]\right\},\tag{8}$$

and

$$F(I_n)\geq\max_{1\leq i\leq n}\left\{w_i\left[f(a)+f(b)-f\left(x_i\right)-f\left(a+b-x_i\right)\right]\right\}.\tag{9}$$

If $\mathbf{w}$ satisfy (2) and $A_{I_n}\left(\mathbf{x},\mathbf{w}\right)\in[a,b]$, then

$$F(I_n)\leq\min_{2\leq j\leq n}\left\{(w_1+w_j)\left[f(a)+f(b)-\frac{w_1f\left(x_1\right)+w_jf\left(x_j\right)}{w_1+w_j}-f\left(a+b-\frac{w_1x_1+w_jx_j}{w_1+w_j}\right)\right]\right\}.\tag{10}$$

Proof. Suppose that $w_i>0$ for all $i=1,\ldots,n$. Similarly as $F(I_n)\geq F(I_2)$ in Corollary 2.5, we may conclude that

$$F(I_n)\geq F\left(\{i,j\}\right)\ \text{ for all } i\neq j\in\{1,\ldots,n\},\tag{11}$$

so the inequality (8) immediately follows. From (11) we have $F(I_n)\geq F\left(\{i\}\right)$ for all $i\in\{1,\ldots,n\}$, so the inequality (9) is also proved.

The inequality (10) can be proved in the similar way. □

Remark 2.7. Analogous assertions can be formulated for concave functions using the fact that f is concave iff $-f$ is convex. □

3. APPLICATIONS

Let A_n, G_n, H_n, and $M_n^{[r]}$ be the arithmetic, geometric, harmonic, and power mean of order r, respectively, of the real numbers $x_i \in [a,b]$, where $0 < a < b$, formed with the positive weights w_i $(i = 1, \dots, n)$. For the various properties of these means and relations among them we refer the reader to [2]. For example, it is well known that

$$\left(\frac{A_n}{G_n}\right)^{W_n} \geq \left(\frac{A_{n-1}}{G_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \left(\frac{A_1}{G_1}\right)^{W_1} \geq 1,$$
$$W_n\left(A_n - G_n\right) \geq W_{n-1}\left(A_{n-1} - G_{n-1}\right) \geq \cdots \geq W_1\left(A_1 - G_1\right) \geq 0.$$

If we define

$$\widetilde{A}_n := a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i = a + b - A_n,$$

$$\widetilde{G}_n := \frac{ab}{\left(\prod_{i=1}^{n} x_i^{w_i}\right)^{\frac{1}{W_n}}} = \frac{ab}{G_n},$$

$$\widetilde{H}_n := \left(a^{-1} + b^{-1} - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^{-1}\right)^{-1} = \left(a^{-1} + b^{-1} - H_n^{-1}\right)^{-1},$$

$$\widetilde{M}_n^{[r]} := \left(a^r + b^r - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^r\right)^{\frac{1}{r}} = \left(a^r + b^r - \left(M_n^{[r]}\right)^r\right)^{\frac{1}{r}},$$

then we have the following results.

Theorem 3.1. (i)

$$\left(\frac{\widetilde{A}_n}{\widetilde{G}_n}\right)^{W_n} \geq \left(\frac{\widetilde{A}_{n-1}}{\widetilde{G}_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \left(\frac{\widetilde{A}_1}{\widetilde{G}_1}\right)^{W_1} \geq 1, \tag{12}$$

(ii)

$$W_n\left(\widetilde{A}_n - \widetilde{G}_n\right) \geq W_{n-1}\left(\widetilde{A}_{n-1} - \widetilde{G}_{n-1}\right) \geq \cdots \geq W_1\left(\widetilde{A}_1 - \widetilde{G}_1\right) \geq 0.$$

Proof. (i) Applying Corollary 2.5 to the convex function $f(x) = -\ln x$, we obtain

$$\ln\left(\frac{\widetilde{A}_n}{\widetilde{G}_n}\right)^{W_n} \geq \ln\left(\frac{\widetilde{A}_{n-1}}{\widetilde{G}_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \ln\left(\frac{\widetilde{A}_1}{\widetilde{G}_1}\right)^{W_1} \geq 0,$$

from which (12) follows.

(ii) Applying Corollary 2.5 to the convex function $f(x) = \exp x$, and replacing a, b, and x_i with $\ln a$, $\ln b$, and $\ln x_i$ respectively, we obtain

$$W_n\left(\widetilde{A}_n - \widetilde{G}_n\right) \geq W_{n-1}\left(\widetilde{A}_{n-1} - \widetilde{G}_{n-1}\right) \geq \cdots \geq W_1\left(\widetilde{A}_1 - \widetilde{G}_1\right) \geq 0,$$

since in this case

$$F(I_k) = W_k\left(a+b-\frac{1}{W_k}\sum_{i=1}^{k} w_i x_i - \exp\left(\ln a + \ln b - \frac{1}{W_k}\sum_{i=1}^{k} w_i \ln x_i\right)\right)$$
$$= W_k\left(\widetilde{A}_k - \widetilde{G}_k\right). \qquad \square$$

Corollary 3.2. (i)

$$\left(\frac{\widetilde{G}_n}{\widetilde{H}_n}\right)^{W_n} \geq \left(\frac{\widetilde{G}_{n-1}}{\widetilde{H}_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \left(\frac{\widetilde{G}_1}{\widetilde{H}_1}\right)^{W_1} \geq 1,$$

(ii)

$$W_n\left(\frac{1}{\widetilde{H}_n} - \frac{1}{\widetilde{G}_n}\right) \geq W_n\left(\frac{1}{\widetilde{H}_{n-1}} - \frac{1}{\widetilde{G}_{n-1}}\right) \geq \cdots \geq W_1\left(\frac{1}{\widetilde{H}_1} - \frac{1}{\widetilde{G}_1}\right) \geq 0.$$

Proof. Directly from Theorem 3.1 by the substitutions $a \to \frac{1}{a}$, $b \to \frac{1}{b}$, $x_i \to \frac{1}{x_i}$. $\square$

Theorem 3.3. *For $r \leq 1$,*

$$W_n\left(\widetilde{A}_n - \widetilde{M}_n^{[r]}\right) \geq W_{n-1}\left(\widetilde{A}_{n-1} - \widetilde{M}_{n-1}^{[r]}\right) \geq \cdots \geq W_1\left(\widetilde{A}_1 - \widetilde{M}_1^{[r]}\right) \geq 0. \tag{13}$$

For $r \geq 1$, the inequalities (13) *are reversed.*

Proof. Suppose that $r \leq 1$. Applying Corollary 2.5 to the convex function $f(x) = x^{\frac{1}{r}}$, and replacing a, b, and x_i with a^r, b^r, and x_i^r respectively, we obtain (13) since in this case

$$F(I_k) = W_k\left(a+b-\frac{1}{W_k}\sum_{i=1}^{k} w_i x_i - \left(a^r + b^r - \frac{1}{W_k}\sum_{i=1}^{k} w_i x_i^r\right)^{\frac{1}{r}}\right)$$
$$= W_k\left(\widetilde{A}_k - \widetilde{M}_k^{[r]}\right).$$

If $r \geq 1$, then the function $f(x) = x^{\frac{1}{r}}$ is concave, so the inequalities (13) are reversed. $\square$

Corollary 3.4.

$$W_n\left(\widetilde{A}_n - \widetilde{H}_n\right) \geq W_{n-1}\left(\widetilde{A}_{n-1} - \widetilde{H}_{n-1}\right) \geq \cdots \geq W_1\left(\widetilde{A}_1 - \widetilde{H}_1\right) \geq 0.$$

Remark 3.5. Obviously, the assertion (ii) from Theorem 3.1 is also direct consequence of Theorem 3.3. $\square$

Theorem 3.6. *Let $r, s \in \mathbb{R}$, $r \leq s$.*

(i) *If $s > 0$, then*

$$W_n\left(\left(\widetilde{M}_n^{[s]}\right)^s - \left(\widetilde{M}_n^{[r]}\right)^s\right) \geq W_{n-1}\left(\left(\widetilde{M}_{n-1}^{[s]}\right)^s - \left(\widetilde{M}_{n-1}^{[r]}\right)^s\right) \tag{14}$$
$$\geq \cdots \geq W_1\left(\left(\widetilde{M}_1^{[s]}\right)^s - \left(\widetilde{M}_1^{[r]}\right)^s\right) \geq 0,$$

(ii) *If $s < 0$, then the inequalities* (14) *are reversed.*

Proof. Suppose that $s > 0$. Applying Corollary 2.5 to the convex function $f(x) = x^{\frac{s}{r}}$, and replacing a, b, and x_i with a^r, b^r, and x_i^r respectively, we obtain (14) since

$$\begin{aligned} F(I_k) &= W_k \left(a^s + b^s - \frac{1}{W_k} \sum_{i=1}^{k} w_i x_i^s - \left(a^r + b^r - \frac{1}{W_k} \sum_{i=1}^{k} w_i x_i^r \right)^{\frac{s}{r}} \right) \\ &= W_k \left(\left(\widetilde{M}_k^{[s]} \right)^s - \left(\widetilde{M}_k^{[r]} \right)^s \right). \end{aligned}$$

If $s < 0$, then the function $f(x) = x^{\frac{s}{r}}$ is concave, so (14) is reversed. □

Let $\varphi : [a, b] \to \mathbb{R}$ be a strictly monotonic and continuous function, where $[a, b] \subset \mathbb{R}$. Then for a given n-tuple $\mathbf{x} = (x_1, \ldots, x_n) \in [a, b]^n$ and positive n-tuple $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$, the value

$$M_{\varphi}^{[n]} = \varphi^{-1} \left(\frac{1}{W_n} \sum_{i=1}^{n} w_i \varphi(x_i) \right)$$

is well defined and is called *quasi-arithmetic mean* of $\mathbf{x}$ with weights $\mathbf{w}$ (see for example [2, p. 215]). If we define

$$\widetilde{M}_{\varphi}^{[n]} := \varphi^{-1} \left(\varphi(a) + \varphi(b) - \frac{1}{W_n} \sum_{i=1}^{n} w_i \varphi(x_i) \right),$$

then we have the following results.

THEOREM 3.7. *Let $\varphi, \psi : [a, b] \to \mathbb{R}$ be strictly monotonic and continuous functions. If $\psi \circ \varphi^{-1}$ is convex on $[a, b]$, then*

$$\begin{aligned} W_n \left(\psi \left(\widetilde{M}_{\psi}^{[n]} \right) - \psi \left(\widetilde{M}_{\varphi}^{[n]} \right) \right) &\geq W_{n-1} \left(\psi \left(\widetilde{M}_{\psi}^{[n-1]} \right) - \psi \left(\widetilde{M}_{\varphi}^{[n-1]} \right) \right) \\ &\geq \cdots \geq W_1 \left(\psi \left(\widetilde{M}_{\psi}^{[1]} \right) - \psi \left(\widetilde{M}_{\varphi}^{[1]} \right) \right) \geq 0. \end{aligned} \tag{15}$$

If $\psi \circ \varphi^{-1}$ is concave on $[a, b]$, then the inequalities (15) *are reversed.*

Proof. Applying Corollary 2.5 to the convex function $f = \psi \circ \varphi^{-1}$, and replacing a, b, and x_i with $\varphi(a)$, $\varphi(b)$, and $\varphi(x_i)$ respectively, we obtain (15) since in this case

$$\begin{aligned} F(I_k) = W_k \Bigg(&\psi(a) + \psi(b) - \frac{1}{W_k} \sum_{i=1}^{k} w_i \psi(x_i) \\ &- \left(\psi \circ \varphi^{-1} \right) \left(\varphi(a) + \varphi(b) - \frac{1}{W_k} \sum_{i=1}^{k} w_i \varphi(x_i) \right) \Bigg) \\ = W_k &\left(\psi \left(\widetilde{M}_{\psi}^{[k]} \right) - \psi \left(\widetilde{M}_{\varphi}^{[k]} \right) \right). \end{aligned}$$

□

REMARK 3.8. Theorems 3.1, 3.3 and 3.6 follow from Theorem 3.7, by choosing adequate functions φ and ψ, and appropriate substitutions. □

Corollary 3.9. *Let $\varphi, \psi : [a,b] \to \mathbb{R}$ be strictly monotonic and continuous functions. If $\psi \circ \varphi^{-1}$ is convex on $[a,b]$, then*

$$\begin{aligned} & W_n \left(\psi \left(\widetilde{M}_{\psi}^{[n]} \right) - \psi \left(\widetilde{M}_{\varphi}^{[n]} \right) \right) \\ (16) \qquad & \geq \max_{1 \leq i < j \leq n} \left\{ (w_i + w_j) \left[\psi(a) + \psi(b) - \frac{w_i \psi(x_i) + w_j \psi(x_j)}{w_i + w_j} \right. \right. \\ & \left. \left. - \left(\psi \circ \varphi^{-1} \right) \left(\varphi(a) + \varphi(b) - \frac{w_i \varphi(x_i) + w_j \varphi(x_j)}{w_i + w_j} \right) \right] \right\}, \end{aligned}$$

and

(17)

$$\begin{aligned} & W_n \left(\psi \left(\widetilde{M}_{\psi}^{[n]} \right) - \psi \left(\widetilde{M}_{\varphi}^{[n]} \right) \right) \\ & \geq \max_{1 \leq i \leq n} \left\{ w_i \left[\psi(a) + \psi(b) - \psi(x_i) - \left(\psi \circ \varphi^{-1} \right) \left(\varphi(a) + \varphi(b) - \varphi(x_i) \right) \right] \right\}. \end{aligned}$$

If $\psi \circ \varphi^{-1}$ is concave on $[a,b]$, then the inequalities (16) *and* (17)*, with* max *replaced by* min*, are reversed.*

Remark 3.10. Analogous assertions can be formulated for the means of Mercer's type formed with weights satisfying (2). □

4. FURTHER GENERALIZATION

Let E be a nonempty set, $\mathcal{A}$ be an algebra of subsets of E, and L be a linear class of real valued functions $f : E \to \mathbb{R}$ having the properties:

L1: $f, g \in L \Rightarrow (\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

L2: $1 \in L$, i.e., if $f(t) = 1$ for $t \in E$, then $f \in L$;

L3: $f \in L, E_1 \in \mathcal{A} \Rightarrow f \cdot \chi_{E_1} \in L$,

where χ_{E_1} is the indicator function of E_1. It follows from L2, L3 that $\chi_{E_1} \in L$ for every $E_1 \in \mathcal{A}$.

Let $A : L \to \mathbb{R}$ be an isotonic linear functional having the properties:

A1: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$, $\alpha, \beta \in \mathbb{R}$;

A2: $f \in L, f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$;

A3: $A(1) = 1$.

It follows from L3 that for every $E_1 \in \mathcal{A}$ such that $A(\chi_{E_1}) > 0$, the functional A_1 defined for all $f \in L$ as $\frac{A(f \cdot \chi_{E_1})}{A(\chi_{E_1})}$ is an isotonic linear functional with $A_1(1) = 1$. Furthermore, we observe that

$$A(\chi_{E_1}) + A\left(\chi_{E \setminus E_1}\right) = 1, \tag{18}$$

$$A(f) = A(f \cdot \chi_{E_1}) + A(f \cdot \chi_{E \setminus E_1}). \tag{19}$$

Let $\varphi : [a,b] \to \mathbb{R}$ be a continuous function, where $[a,b] \subset \mathbb{R}$. In [1], under the above assumptions, the following variant of the Jessen's inequality is proved: if φ is convex, then

$$\varphi\left(a+b-A\left(f\right)\right) \leq \varphi\left(a\right)+\varphi\left(b\right)-A\left(\varphi\left(f\right)\right); \tag{20}$$

if φ is concave, then the inequality (20) is reversed.

For $E_1 \in \mathcal{A}$ and $f \in L$ such that $\varphi(f) \in L$, we define the function $F_f : \mathcal{A} \to \mathbb{R}$ with

$$F_f\left(E_1\right) = A\left(\chi_{E_1}\right)\left[\varphi\left(a\right)+\varphi\left(b\right)-\frac{A(\varphi\left(f\right)\cdot\chi_{E_1})}{A\left(\chi_{E_1}\right)}-\varphi\left(a+b-\frac{A(f\cdot\chi_{E_1})}{A\left(\chi_{E_1}\right)}\right)\right].$$

THEOREM 4.1. *Under the above assumptions, if φ is convex, then*

$$F_f\left(E\right) \geq F_f\left(E_1\right)+F_f\left(E\setminus E_1\right) \geq F_f\left(E_1\right) \geq 0 \tag{21}$$

for all $E_1 \in \mathcal{A}$ such that $0 < A\left(\chi_{E_1}\right) < 1$.

Proof. Since φ is continuous and convex, the same is also true for the function

$$\psi : [a,b] \to \mathbb{R}$$

defined as

$$\psi(t) = \varphi(a+b-t), \quad t \in [a,b].$$

Hence, the following inequality holds for every $t_1, t_2 \in [a,b]$ and $p,q \in \mathbb{R}$ such that $p+q=1$

$$p\psi\left(t_1\right)+q\psi\left(t_2\right) \geq \psi\left(pt_1+qt_2\right),$$

i.e.,

$$p\varphi(a+b-t_1)+q\varphi(a+b-t_2) \geq \varphi(a+b-(pt_1+qt_2)).$$

If we let $p = A\left(\chi_{E_1}\right)$, $q = A\left(\chi_{E\setminus E_1}\right)$, $t_1 = \frac{A(f\cdot\chi_{E_1})}{A\left(\chi_{E_1}\right)}$, and $t_2 = \frac{A(f\cdot\chi_{E\setminus E_1})}{A\left(\chi_{E\setminus E_1}\right)}$, then $p+q=1$ by the equality (18), and $pt_1+qt_2 = A(f)$ by the equality (19). Similarly, we can use that

$$A(\varphi\left(f\right)) = A(\varphi\left(f\right)\cdot\chi_{E_1})+A(\varphi\left(f\right)\cdot\chi_{E\setminus E_1}).$$

Hence, we have

$$A\left(\chi_{E_1}\right)\varphi\left(a+b-\frac{A(f\cdot\chi_{E_1})}{A\left(\chi_{E_1}\right)}\right)+A\left(\chi_{E\setminus E_1}\right)\varphi\left(a+b-\frac{A(f\cdot\chi_{E\setminus E_1})}{A\left(\chi_{E\setminus E_1}\right)}\right)$$
$$\geq \varphi\left(a+b-A(f)\right).$$

Multiplying the above inequality with (-1) and adding to the both sides the term

$$\varphi\left(a\right)+\varphi\left(b\right)-A(\varphi\left(f\right))$$

we obtain the first inequality in (21).

To prove the remaining two inequalities in (21), observe that they are simple consequences of (20) applied to the isotonic linear functional A_1 defined for $f \in L$ as $\frac{A(f\cdot\chi_{E_1})}{A\left(\chi_{E_1}\right)}$ and having $A_1\left(1\right)=1$. □

Corollary 4.2. *Let* $\{E_1, \ldots, E_n\}$ *be a partition of* E *(i.e.,* $E = \bigcup_{j=1}^{n} E_j, E_i \cap E_j = \emptyset$ *for all* $i \neq j \in \{1, \ldots, n\}$*) such that* $0 < A\left(\chi_{E_j}\right) < 1$ *for all* $j \in \{1, \ldots, n\}$*. If* φ *is convex, then*

$$F_f(E) \geq \sum_{j=1}^{n} F_f(E_j), \tag{22}$$

and

$$\begin{aligned} F_f(E) &\geq F_f\left(\bigcup_{j=1}^{n-1} E_j\right) \geq F_f\left(\bigcup_{j=1}^{n-2} E_j\right) \\ &\geq \cdots \geq F_f(E_1 \cup E_2) \geq F_f(E_1) \geq 0. \end{aligned} \tag{23}$$

Proof. Directly from Theorem 4.1 by induction. □

Remark 4.3. If φ is concave, then the inequalities (21)–(23) are reversed. □

Corollary 4.4. *Let* $\{E_1, \ldots, E_n\}$ *be a partition of* E *such that* $0 < A\left(\chi_{E_j}\right) < 1$ *for all* $j \in \{1, \ldots, n\}$*. If* φ *is convex, then*

$$F_f(E) \geq \max_{1 \leq i < j \leq n} \{F_f(E_i \cup E_j)\}, \tag{24}$$

and

$$F_f(E) \geq \max_{1 \leq j \leq n} \{F_f(E_j)\}. \tag{25}$$

If φ *is concave, then the inequalities* (24) *and* (25)*, with* max *replaced by* min*, are reversed.*

Remark 4.5. We also may obtain similar results as in Theorem 3.7 and Corollary 3.9, for the generalized quasi-arithmetic means of Mercer's type defined in [1], as

$$\widetilde{M}_\varphi(f, A) = \varphi^{-1}(\varphi(a) + \varphi(b) - A(\varphi(f))). \quad \square$$

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