DIFFERENTIABILITY WITH RESPECT TO A PARAMETER FOR A LOTKA-VOLTERRA SYSTEM WITH DELAYS, VIA STEP METHOD

DIANA OTROCOL

Abstract. In this paper, using the step method, we establish the differentiability with respect to parameter for a Lotka-Volterra system with two delays.

MSC 2000. 34L05, 47H10.

Keywords. Differential equations, delay, step method.

1. INTRODUCTION

The purpose of this paper is to study the $\lambda$-dependence of the solution of the Lotka-Volterra problem:

\begin{align}
    x_1'(t) &= f_i(t, x_1(t), x_2(t), x_1(t - \tau_1), x_2(t - \tau_2); \lambda), \quad t \in [t_0, b], \quad \lambda \in \mathcal{J}, \\
    x_1(t) &= \varphi(t), \quad t \in [t_0 - \tau_1, t_0], \\
    x_2(t) &= \psi(t), \quad t \in [t_0 - \tau_2, t_0].
\end{align}

There have been many studies on this subject. Differentiability with initial data for the functional differential equations was first established by Hale in [1], but differentiability with respect to delays for delay differential equations was proved by Hale and Ladeira in [2] and by A. Tămășan in [12]. The paper of Hokkanen and Moroșanu [4] gives a proof for delay differential equations case using the step method. The Picard operators technique proposed by I.A. Rus, [8], [9], [10], was used by V. Mureșanu [5] to prove continuity with respect to $\lambda$, M. Șerban [11] and us [7] using the theorem of fibre contraction.

In this paper we use the following theorem for the simple case of ordinary differential system.

Theorem 1. [3] Consider the initial value problem

\begin{equation}
    x'(t) = g(t, x(t); \lambda), \quad t \in [a, b], \\
    x(t_0) = x_0,
\end{equation}

*This work has been supported by MEdC-ANCS under grant ET 3233/17.10.2005.

†“Tiberiu Popoviciu” Institute of Numerical Analysis, P.O. Box. 68-1, Cluj-Napoca, Romania, e-mail: dotrocol@ictp.acad.ro.
where \( g \in C^1([a, b] \times \mathbb{R} \times J, \mathbb{R}) \), \( \left\| \frac{\partial g(t, u, \lambda)}{\partial u} \right\|_{\mathbb{R}} \leq M_1 \). Then the unique solution \( x \in C^1([a, b] \times J) \).

2. MAIN RESULT

We consider the following Lotka-Volterra system with parameter:

\[
\begin{align*}
(3) & \quad x_1'(t) = f_i(t, x_1(t), x_2(t), x_1(t - \tau_1), x_2(t - \tau_2); \lambda), \quad t \in [t_0, b], \ \lambda \in J, \\
(4) & \quad x_1(t) = \varphi(t), \quad t \in [t_0 - \tau_1, t_0], \\
(5) & \quad x_2(t) = \psi(t), \quad t \in [t_0 - \tau_2, t_0].
\end{align*}
\]

We suppose that:

- \((H_1)\) \( \tau_1 \leq \tau_2; \ t_0 < b; \ J \subset \mathbb{R}, \) a compact interval;
- \((H_2)\) \( f_i \in C^1([t_0, b] \times \mathbb{R}^4 \times J), \ i = 1, 2; \)
- \((H_3)\) \( \exists M_1 > 0 \) such as \( \left\| \frac{\partial f_i}{\partial u_j}(t, u_1, u_2, u_3, u_4; \lambda) \right\| \leq M_1, \)
  - for all \( t \in [t_0, b], \ u_j \in \mathbb{R}, \ j = 1, 2, \lambda \in J, \ i = 1, 2; \)
- \((H_4)\) \( \varphi \in C([t_0 - \tau_1, t_0]), \ \psi \in C([t_0 - \tau_2, t_0]). \)

In the above conditions, from Theorem 1 in \([6]\) we have that the problem \((3)-(4)\) has a unique solution, \((x_1(t; \lambda), x_2(t; \lambda))\).

We prove that

\[
\begin{align*}
x_1(t; \cdot) & \in C^1(J), \text{ for all } t \in [t_0 - \tau_1, b], \\
x_2(t; \cdot) & \in C^1(J), \text{ for all } t \in [t_0 - \tau_2, b],
\end{align*}
\]

applying Theorem \([1]\) and using the step method.

We consider the system

\[
(5) \quad x_1'(t, \lambda) = f_i(t, x_1(t, \lambda), x_2(t, \lambda), x_1(t - \tau_1, \lambda), x_2(t - \tau_2, \lambda); \lambda),
\quad t \in [t_0, b], \ \lambda \in J
\]

with the initial conditions

\[
(6) \quad x_1(t; \lambda) = \varphi(t; \lambda),
\quad x_2(t; \lambda) = \psi(t; \lambda),
\]

where \( x_1 \in C([t_0 - \tau_1, b] \times J) \cap C^1[t_0, b], \) \( x_2 \in C([t_0 - \tau_2, b] \times J) \cap C^1[t_0, b]. \)

**Theorem 2.** In conditions \((H_1), (H_2), (H_4),\) the solution \((x_1(t; \lambda), x_2(t; \lambda))\) of the problem \((5)-(6)\) is continuously with respect to \( \lambda.\)

**Theorem 3.** In conditions \((H_1)-(H_3),\) the solution \((x_1(t; \lambda), x_2(t; \lambda))\) of the problem \((5)-(6)\) is differentiable with respect to \( \lambda.\)

**Proof.** We prove that

\[
\begin{align*}
x_1(t; \cdot) & \in C^1(J), \text{ for all } t \in [t_0 - \tau_1, b], \\
x_2(t; \cdot) & \in C^1(J), \text{ for all } t \in [t_0 - \tau_2, b],
\end{align*}
\]

applying Theorem \([1]\) and using the step method.

For \( t \in [t_0, t_0 + \tau_1], \) from Theorem \([1]\) we have that \( x_1(t; \cdot) \in C^1(J), \) \( x_2(t; \cdot) \in C^1(J).\)
For \( t \in [t_0 + \tau_1, t_0 + 2\tau_1] \), from Theorem 1 we have that \( x_1(t; \cdot) \in C^1(J) \), \( x_2(t; \cdot) \in C^1(J) \). From (7) and (H2) it follows that \( x_1'(t_0 + \tau_1 - 0) = x_1'(t_0 + \tau_1 + 0) \) and \( x_2'(t_0 + \tau_1 - 0) = x_2'(t_0 + \tau_1 + 0) \). Then \((x_1(t; \lambda), x_2(t; \lambda))\) is differentiable at \( t_0 + \tau_1 \).

For \( t \in [t_0 + n\tau_1, t_0 + 2n\tau_1] \), from Theorem 1 we have that \( x_1(t; \cdot) \in C^1(J) \), \( x_2(t; \cdot) \in C^1(J) \). From (5) and (H2) it follows that \( x_1'(t_0 + n\tau_1 - 0) = x_1'(t_0 + n\tau_1 + 0) \) and \( x_2'(t_0 + n\tau_1 - 0) = x_2'(t_0 + n\tau_1 + 0) \). Then \((x_1(t; \lambda), x_2(t; \lambda))\) is differentiable at \( t_0 + n\tau_1 \).

So \((x_1(t; \lambda), x_2(t; \lambda))\) is \( C^1 \) at the knots.

We present below a simple example to illustrate the procedures of applying our results.

**Example 1.** Consider the Lotka-Volterra type predator-prey system with two delays with parameter:

\[
\begin{align*}
x_1'(t, \lambda) &= \lambda x_1(t, \lambda) [2 - x_1(t, \lambda) - x_2(t, \lambda)] \\
x_2'(t, \lambda) &= \lambda x_2(t, \lambda) [2 - x_1(t - 1, \lambda) - x_2(t - 2, \lambda)],
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
x_1(t; \lambda) &= t + 1, \quad t \in [t_0 - 1, t_0], \\
x_2(t; \lambda) &= t + 2, \quad t \in [t_0 - 2, t_0].
\end{align*}
\]

Equation (7) is of the form (5). Therefore, the assumptions \((H_1) - (H_4)\) are satisfied by system (7)–(5). Moreover, Theorem 2 holds. Applying Theorem 3 we draw the following conclusions.

For \( t \in [t_0, t_0 + 1] \), from Theorem 1 we have that \( x_1(t; \cdot) \in C^1(J) \), \( x_2(t; \cdot) \in C^1(J) \).

![Fig. 1. Comparison between Lotka-Volterra systems with different parameters.](image)

For \( t \in [t_0 + 1, t_0 + 2] \), from Theorem 1 we have that \( x_1(t; \cdot) \in C^1(J) \), \( x_2(t; \cdot) \in C^1(J) \). From (5) and (H2) it follows that \( x_1'(t_0 + 1 - 0) = x_1'(t_0 + 1 + 0) \) and \( x_2'(t_0 + 1 - 0) = x_2'(t_0 + 1 + 0) \). Then \((x_1(t; \lambda), x_2(t; \lambda))\) is differentiable
at \( t_0 + 1 \). So \((x_1(t; \lambda), x_2(t; \lambda))\) is \( C^1 \) at the knots. Therefore Theorem 3 is applicable.

Here we give a portrait of the trajectory of (7)–(8) drawn using MATLAB facilities. The results from numerical computation are plotted for \( \lambda = 1.35, 1.37, 1.39, 1.41 \) in Fig. 1.

REFERENCES


Received by the editors: March 16, 2006.