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DIFFERENTIABILITY WITH RESPECT TO A PARAMETER FOR A LOTKA-VOLTERRA SYSTEM WITH DELAYS, VIA STEP METHOD*

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Abstract. In this paper, using the step method, we establish the differentiability with respect to parameter for a Lotka-Volterra system with two delays.

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1. INTRODUCTION

The purpose of this paper is to study the λ-dependence of the solution of the Lotka-Volterra problem:

$$x_i'(t) = f_i(t, x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2); \lambda), \qquad t \in [t_0, b],\ \lambda \in J \tag{1}$$

$$\begin{aligned} x_1(t) &= \varphi(t),\ t \in [t_0-\tau_1, t_0], \\ x_2(t) &= \psi(t),\ t \in [t_0-\tau_2, t_0]. \end{aligned} \tag{2}$$

There have been many studies on this subject. Differentiability with initial data for the functional differential equations was first established by Hale in [1], but differentiability with respect to delays for delay differential equations was proved by Hale and Ladeira in [2] and by A. Tămăşan in [12]. The paper of Hokkanen and Moroşanu [4] gives a proof for delay differential equations case using the step method. The Picard operators technique proposed by I.A. Rus, [8], [9], [10], was used by V. Mureşanu [5] to prove continuity with respect to λ, M. Şerban [11] and us [7] using the theorem of fibre contraction.

In this paper we use the following theorem for the simple case of ordinary differential system.

Theorem 1. [3] *Consider the initial value problem*

$$\begin{aligned} &x'(t) = g(t, x(t); \lambda), \quad t \in [a, b], \\ &x(t_0) = x_0, \end{aligned}$$

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where $g \in C^1([a,b] \times \mathbb{R} \times J, \mathbb{R})$, $\|\frac{\partial g(t,u,\lambda)}{\partial u}\|_{\mathbb{R}} \leq M_1$. *Then the unique solution* $x \in C^1([a,b] \times J)$.

2. MAIN RESULT

We consider the following Lotka-Volterra system with parameter:

$$x_i'(t) = f_i(t, x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2); \lambda), \qquad t \in [t_0, b], \ \lambda \in J, \tag{3}$$

$$\begin{aligned} x_1(t) &= \varphi(t), \ t \in [t_0 - \tau_1, t_0], \\ x_2(t) &= \psi(t), \ t \in [t_0 - \tau_2, t_0]. \end{aligned} \tag{4}$$

We suppose that:

- (H_1) $\tau_1 \leq \tau_2$; $t_0 < b$; $J \subset \mathbb{R}$, a compact interval;
- (H_2) $f_i \in C^1([t_0, b] \times \mathbb{R}^4 \times J)$, $i = 1, 2$;
- (H_3) $\exists M_1 > 0$ such as $\|\frac{\partial f_i}{\partial u_j}(t, u_1, u_2, u_3, u_4; \lambda)\|_{\mathbb{R}} \leq M_1$, for all $t \in [t_0, b]$, $u_j \in \mathbb{R}$, $j = \overline{1,4}$, $\lambda \in J$, $i = 1, 2$;
- (H_4) $\varphi \in C([t_0 - \tau_1, t_0])$, $\psi \in C([t_0 - \tau_2, t_0])$.

In the above conditions, from Theorem 1 in [6] we have that the problem (3)–(4) has a unique solution, $(x_1(t;\lambda), x_2(t;\lambda))$.

We prove that

$$x_1(t;\cdot) \in C^1(J), \text{ for all } t \in [t_0 - \tau_1, b],$$
$$x_2(t;\cdot) \in C^1(J), \text{ for all } t \in [t_0 - \tau_2, b],$$

applying Theorem 1 and using the step method.

We consider the system

$$\begin{gathered} x_i'(t,\lambda) = f_i(t, x_1(t,\lambda), x_2(t,\lambda), x_1(t-\tau_1,\lambda), x_2(t-\tau_2,\lambda); \lambda), \\ t \in [t_0, b], \ \lambda \in J \end{gathered} \tag{5}$$

with the initial conditions

$$\begin{aligned} x_1(t;\lambda) &= \varphi(t;\lambda), \\ x_2(t;\lambda) &= \psi(t;\lambda), \end{aligned} \tag{6}$$

where $x_1 \in C([t_0 - \tau_1, b] \times J) \cap C^1[t_0, b]$, $x_2 \in C([t_0 - \tau_2, b] \times J) \cap C^1[t_0, b]$.

THEOREM 2. *In conditions* (H_1), (H_2), (H_4), *the solution* $(x_1(t;\lambda), x_2(t;\lambda))$ *of the problem* (5)–(6) *is continuosly with respect to* λ.

THEOREM 3. *In conditions* (H_1)–(H_4), *the solution* $(x_1(t;\lambda), x_2(t;\lambda))$ *of the problem* (5)–(6) *is differentiable with respect to* λ.

Proof. We prove that

$$x_1(t;\cdot) \in C^1(J), \text{ for all } t \in [t_0 - \tau_1, b],$$
$$x_2(t;\cdot) \in C^1(J), \text{ for all } t \in [t_0 - \tau_2, b],$$

applying Theorem 1 and using the step method.

For $t \in [t_0, t_0 + \tau_1]$, from Theorem 1 we have that $x_1(t;\cdot) \in C^1(J)$, $x_2(t;\cdot) \in C^1(J)$.

For $t \in [t_0 + \tau_1, t_0 + 2\tau_1]$, from Theorem 1 we have that $x_1(t; \cdot) \in C^1(J)$, $x_2(t; \cdot) \in C^1(J)$. From (5) and (H_2) it follows that $x_1'(t_0+\tau_1-0) = x_1'(t_0+\tau_1+0)$ and $x_2'(t_0+\tau_1-0) = x_2'(t_0+\tau_1+0)$. Then $(x_1(t;\lambda), x_2(t;\lambda))$ is differentiable at $t_0 + \tau_1$.

For $t \in [t_0 + n\tau_1, t_0 + \tau_2]$, from Theorem 1 we have that $x_1(t; \cdot) \in C^1(J)$, $x_2(t; \cdot) \in C^1(J)$. From (5) and (H_2) it follows that $x_1'(t_0 + n\tau_1 - 0) = x_1'(t_0 + n\tau_1 + 0)$ and $x_2'(t_0 + n\tau_1 - 0) = x_2'(t_0 + n\tau_1 + 0)$. Then $(x_1(t;\lambda), x_2(t;\lambda))$ is differentiable at $t_0 + n\tau_1$.

So $(x_1(t;\lambda), x_2(t;\lambda))$ is C^1 at the knots. □

We present below a simple example to illustrate the procedures of applying our results.

EXAMPLE 1. Consider the Lotka-Volterra type predator-prey system with two delays with parameter:

$$\begin{array}{l} x_1'(t,\lambda) = \lambda x_1(t;\lambda)\,[2 - x_1(t;\lambda) - x_2(t;\lambda)] \\ x_2'(t,\lambda) = \lambda x_2(t;\lambda)\,[2 - x_1(t-1;\lambda) - x_2(t-2;\lambda)] \end{array}, \quad t \in [0,2] \tag{7}$$

with the initial conditions

$$\begin{array}{ll} x_1(t;\lambda) = t + 1, & t \in [t_0 - 1, t_0], \\ x_2(t;\lambda) = t + 2, & t \in [t_0 - 2, t_0]. \end{array} \tag{8}$$ □

Equation (7) is of the form (5). Therefore, the assumptions (H_1)–(H_4) are satisfied by system (7)–(8). Moreover, Theorem 2 holds. Applying Theorem 3, we draw the following conclusions.

For $t \in [t_0, t_0 + 1]$, from Theorem 1 we have that $x_1(t; \cdot) \in C^1(J)$, $x_2(t; \cdot) \in C^1(J)$.

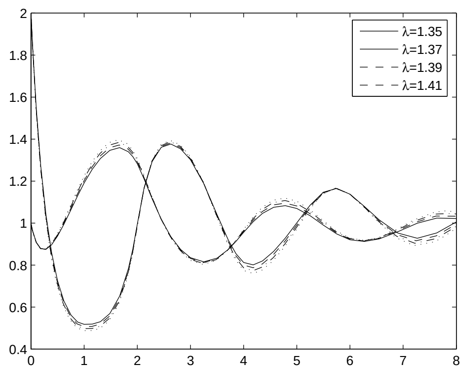

Fig. 1. Comparison between Lotka-Volterra systems with different parameters.

For $t \in [t_0 + 1, t_0 + 2]$, from Theorem 1 we have that $x_1(t; \cdot) \in C^1(J)$, $x_2(t; \cdot) \in C^1(J)$. From (5) and (H_2) it follows that $x_1'(t_0+1-0) = x_1'(t_0+1+0)$ and $x_2'(t_0 + 1 - 0) = x_2'(t_0 + 1 + 0)$. Then $(x_1(t;\lambda), x_2(t;\lambda))$ is differentiable at $t_0 + 1$. So $(x_1(t;\lambda), x_2(t;\lambda))$ is C^1 at the knots. Therefore Theorem 3 is applicable.

Here we give a portrait of the trajectory of (7)–(8) drawn using MATLAB facilities. The results from numerical computation are plotted for $\lambda = 1.35, 1.37, 1.39, 1.41$ in Fig. 1.

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