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ABOUT SOME MEAN-VALUE THEOREMS FOR B-DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper we shall demonstrate a general mean-value theorem for B-differentiable functions. By particularization, we obtain Pompeiu-type and Ivan-type theorems for B-differentiable functions.

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1. PRELIMINARIES

In the following, let X and Y be compact real intervals. The definition of B-differentiability was introduced by K. Bögel in the papers [2] and [3].

DEFINITION 1. *A function $f : X \times Y \to \mathbb{R}$ is called a B-differentiable function in $(x_0, y_0) \in X \times Y$ iff exists and if the limit*

$$\lim_{(x,y)\to(x_0,y_0)} \tfrac{\Delta f[(x,y),(x_0,y_0)]}{(x-x_0)(y-y_0)} \tag{1}$$

is finite, where $\Delta f[(x,y),(x_0,y_0)] = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$ denote a so-called mixed difference of function f.

The limit from (1) is named the B-differential of f in the point (x_0, y_0) and is noted by $D_B f(x_0, y_0)$.

We recall the following results (see [2], [3] or [6]).

THEOREM 2. *Let $f : X \times Y \to \mathbb{R}$ be a function. If f admits the derivatives f'_x, f''_{xy} in a neighborhood of the point $(x_0, y_0) \in X \times Y$ and the derivative f''_{xy} is continuous in (x_0, y_0), then f is B-differentiable in (x_0, y_0) and*

$$D_B f(x_0, y_0) = f''_{xy}(x_0, y_0). \tag{2}$$

THEOREM 3. *Let $f : [a,b]\times[a',b'] \to \mathbb{R}$ be a function. If f is B-differentiable on $[a,b] \times [a',b']$ and*

$$\Delta f[(a,a'),(b,b')] = 0 \tag{3}$$

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then there exists $(\xi,\eta)\in(a,b)\times(a',b')$ *such that*

$$D_B f(\xi,\eta)=0. \tag{4}$$

REMARK 1. This theorem is a Rolle-type theorem. □

THEOREM 4. *Let* $f:[a,b]\times[a',b']\to\mathbb{R}$ *be a function. If* f *is* B*-differentiable on* $[a,b]\times[a',b']$*, then there exists* $(\xi,\eta)\in(a,b)\times(a',b')$ *such that*

$$\Delta f[(a,a'),(b,b')]=(b-a)(b'-a')D_B f(\xi,\eta). \tag{5}$$

REMARK 2. This theorem is a Lagrange-type theorem. □

THEOREM 5. *Let* $f,g:[a,b]\times[a',b']\to\mathbb{R}$ *be two functions. If* f *and* g *are* B*-differentiable on* $[a,b]\times[a',b']$*, then there exists* $(\xi,\eta)\in(a,b)\times(a',b')$ *such that*

$$\Delta f[(a,a'),(b,b')]D_B g(\xi,\eta)=\Delta g[(a,a'),(b,b')]D_B f(\xi,\eta). \tag{6}$$

REMARK 3. This theorem is a Cauchy-type theorem. □

2. MAIN RESULTS

LEMMA 6. *If* $d\notin[a,b]$ *and* $d'\notin[a',b']$*, then*

$$\Delta\frac{1}{(\cdot-d)(*-d')}\,[(a,a'),(b,b')]=\frac{(a-b)(a'-b')}{(a-d)(b-d)(a'-d')(b'-d')}\,, \tag{7}$$

where "$\cdot$" and "$*$" stand for the first and the second variable.

Proof. Follows immediately. □

THEOREM 7. *Let* $f:[a,b]\times[a',b']\to\mathbb{R}$ *be a* B*-differentiable function on* $[a,b]\times[a',b']$ *and* $d\notin[a,b]$, $d'\notin[a',b']$*. Then there exists a point* $(\xi,\eta)\in(a,b)\times(a',b')$ *such that*

$$\begin{aligned}\Delta\frac{1}{(\cdot-d)(*-d')}\,[(a,a'),(b,b')]D_B\frac{f(\cdot,*)}{(\cdot-d)(*-d')}\,(\xi,\eta)&=\\ =\Delta\frac{f(\cdot,*)}{(\cdot-d)(*-d')}\,[(a,a'),(b,b')]D_B\frac{1}{(\cdot-d)(*-d')}\,(\xi,\eta).\end{aligned} \tag{8}$$

If in addition f *admits the derivatives* f'_x, f'_y, f''_{xy} *on* $[a,b]\times[a',b']$ *and the derivative* f''_{xy} *is continuous on* $(a,b)\times(a',b')$ *then*

(9)

$$\begin{aligned}&\frac{aa'f(b,b')-a'bf(a,b')-ab'f(b,a')+bb'f(a,a')}{(a-b)(a'-b')}-\xi\eta f''_{xy}(\xi,\eta)+\xi f'x(\xi,\eta)\\&\quad+\eta f'y(\xi,\eta)-f(\xi,\eta)=\\&=(dd'-\xi d'-\eta d)f''_{xy}(\xi,\eta)+df'_x(\xi,\eta)+d'f'y(\xi,\eta)+\\&\quad-\frac{(dd'-ad'-a'd)f(b,b')-(dd'-bd'-a'd)f(a,b')}{(a-b)(a'-b')}+\frac{(dd'-ad'-b'd)f(b,a')-(dd'-b'd-bd')f(a,a')}{(a-b)(a'-b')}.\end{aligned}$$

Proof. We apply Theorem 5 for the functions $F,G:[a,b]\times[a',b']\to\mathbb{R}$, $F(x,y)=\frac{f(x,y)}{(x-d)(y-d')}$, $G(x,y)=\frac{1}{(x-d)(y-d')}$ and we make the calculus. □

Theorem 8. *Let $\varphi : [a,b] \to \mathbb{R}$ be a function. If*
(1) φ is continuous on $[a,b]$,
(2) φ is differentiable on (a,b),
(3) $d \notin [a,b]$,
then there exists $\xi \in (a,b)$ such that

$$\frac{a\varphi(b)-b\varphi(a)}{a-b} - \varphi(\xi) + \xi\varphi'(\xi) = d\left(\varphi'(\xi) - \frac{\varphi(a)-\varphi(b)}{a-b}\right). \tag{10}$$

Proof. In Theorem 7 we take $d' = 0$ and $f(x,y) = \varphi(x)$. □

Remark 4. For another proof, see [9] or [10]. □

Theorem 9. *Let $f : [a,b] \times [a',b'] \to \mathbb{R}$ be a B-differentiable function such that $0 \notin [a,b]$, $0 \notin [a',b']$, f admits the derivatives f'_x, f'_y, f''_{xy} on $[a,b] \times [a',b']$ and the derivative f''_{xy} is continuous on $(a,b) \times (a',b')$. Then there exists $(\xi,\eta) \in (a,b) \times (a',b')$ such that*

$$\begin{aligned} &\frac{aa'f(b,b')-a'bf(a,b')-ab'f(b,a')+bb'f(a,a')}{(a-b)(a'-b')} = \\ &= \xi\eta f''_{xy}(\xi,\eta) - \xi f'_x(\xi,\eta) - \eta f'_y(\xi,\eta) + f(\xi,\eta). \end{aligned} \tag{11}$$

Proof. In Theorem 7 we take $d = 0$, $d' = 0$. □

Remark 5. This theorem is a Pompeiu-type theorem, see [5], [7], [9] or [10]. □

Theorem 10. (Pompeiu's Theorem). *Let $\varphi : [a,b] \to \mathbb{R}$ be a function. If*
(1) φ is continuous on $[a,b]$,
(2) φ is differentiable on (a,b),
(3) $0 \notin [a,b]$, then there exists $\xi \in (a,b)$ such that

$$\frac{a\varphi(b)-b\varphi(a)}{a-b} = \varphi(\xi) - \xi\varphi'(\xi). \tag{12}$$

Proof. In Theorem 9 we take $f : [a,b] \times [a,b] \to \mathbb{R}$, $f(x,y) = \varphi(x)$ or in Theorem 8 we take $d = 0$. □

Theorem 11. *Let $f, g : [a,b] \times [a',b'] \to \mathbb{R}$ be two functions B-differentiable on $[a,b] \times [a',b']$. If $g(x,y) \neq 0$ for any $(x,y) \in [a,b] \times [a',b']$, then there exists $(\xi,\eta) \in (a,b) \times (a',b')$ such that*

$$\begin{aligned} &\Delta_{\frac{f(\cdot,*)}{g(\cdot,*)}}[(a,a'),(b,b')] D_B \frac{*}{g(\cdot,*)}(\xi,\eta) = \\ &= \Delta_{\frac{*}{g(\cdot,*)}}[(a,a'),(b,b')] D_B \frac{f(\cdot,*)}{g(\cdot,*)}(\xi,\eta). \end{aligned} \tag{13}$$

If in addition f, g admit the derivatives f'_x, g'_x, f'_y, g'_y, f''_{xy}, g''_{xy} on $[a,b]\times[a',b']$ and the derivatives f''_{xy}, g''_{xy} are continuous on $(a,b)\times(a',b')$ then

$$\begin{aligned}(14)\quad &\left[\tfrac{f(a,a')}{g(a,a')}-\tfrac{f(b,a')}{g(b,a')}-\tfrac{f(a,b')}{g(a,b')}+\tfrac{f(b,b')}{g(b,b')}\right][2\eta g'_x(\xi,\eta)g'_y(\xi,\eta)-\\ &-g(\xi,\eta)g'_x(\xi,\eta)-\eta g(\xi,\eta)g''_{xy}(\xi,\eta)]=\\ &=\left[\tfrac{a'}{g(a,a')}-\tfrac{a'}{g(b,a')}-\tfrac{b'}{g(a,b')}+\tfrac{b'}{g(b,b')}\right][2f(\xi,\eta)g'_x(\xi,\eta)g'_y(\xi,\eta)-\\ &-g(\xi,\eta)f'_x(\xi,\eta)g'_y(\xi,\eta)-g(\xi,\eta)f'_y(\xi,\eta)g'_x(\xi,\eta)-\\ &-f(\xi,\eta)g(\xi,\eta)g''_{xy}(\xi,\eta)+g^2(\xi,\eta)f''_{xy}(\xi,\eta)].\end{aligned}$$

Proof. We apply Theorem 5 for the functions $F,G:[a,b]\times[a',b']\to\mathbb{R}$, $F(x,y)=\frac{f(x,y)}{g(x,y)}$, $G(x,y)=\frac{y}{g(x,y)}$ and we make the calculus. □

REMARK 6. This theorem is a Boggio-type theorem, see [1]. □

THEOREM 12. (Boggio's Theorem). *Let $\varphi,\psi:[a,b]\to\mathbb{R}$ be two functions satisfying*
(1) *φ, ψ are continuous on $[a,b]$,*
(2) *ψ has no roots in $[a,b]$,*
(3) *ψ' has no roots in (a,b).*
Then there exists $\xi\in(a,b)$ such that

$$(15)\quad \frac{\psi(a)\varphi(b)-\psi(b)\varphi(a)}{\psi(a)-\psi(b)}=\varphi(c)-\psi(c)\frac{\varphi'(c)}{\psi'(c)}.$$

Proof. In Theorem 11 let $f,g:[a,b]\times[a',b']\to\mathbb{R}$, $f(x,y)=\varphi(x)$, $g(x,y)=\psi(x)$. □

THEOREM 13. *Let $f:[a,b]\times[a',b']\to\mathbb{R}$ be a B-differentiable function, $d\in$ such that $f(x,y)\neq d$ for any $(x,y)\in[a,b]\times[a',b']$. Then there exists $(\xi,\eta)\in(a,b)\times(a',b')$ such that*

$$\begin{aligned}(16)\quad &\Delta_{\frac{\cdot\,*}{f(\cdot,*)-d}}[(a,a'),(b,b')]D_B{\tfrac{*}{f(\cdot,*)-d}}(\xi,\eta)=\\ &=\Delta_{\frac{*}{f(\cdot,*)-d}}[(a,a'),(b,b')]D_B{\tfrac{\cdot\,*}{f(\cdot,*)-d}}(\xi,\eta).\end{aligned}$$

If in addition f admits the derivatives f'_x, f'_y, f''_{xy} on $[a,b]\times[a',b']$, f''_{xy} continuous on $(a,b)\times(a',b')$ then

$$\begin{aligned}(17)\quad &\left[\tfrac{aa'}{f(a,a')-d}-\tfrac{ba'}{f(b,a')-d}-\tfrac{ab'}{f(a,b')-d}+\tfrac{bb'}{f(b,b')-d}\right]\cdot\\ &\cdot[2\eta f'_x(\xi,\eta)f'_y(\xi,\eta)-(f(\xi,\eta)-d)f'_x(\xi,\eta)-\eta(f(\xi,\eta)-d)f''_{xy}(\xi,\eta)]=\\ &=\left[\tfrac{a'}{f(a,a')-d}-\tfrac{a'}{f(b,a')-d}-\tfrac{b'}{f(a,b')-d}+\tfrac{b'}{f(b,b')-d}\right]\cdot\\ &\cdot[f^2(\xi,\eta)-\eta f(\xi,\eta)f'_y(\xi,\eta)-\xi f(\xi,\eta)f'_x(\xi,\eta)-\\ &-\xi\eta f(\xi,\eta)f''_{xy}(\xi,\eta)+2\xi\eta f'_x(\xi,\eta)f'_y(\xi,\eta)].\end{aligned}$$

Proof. In Theorem 5 let $F, G : [a,b] \times [a',b'] \to \mathbb{R}$, $F(x,y) = \frac{xy}{f(x,y)-d}$, $G(x,y) = \frac{y}{f(x,y)-d}$. □

REMARK 7. This theorem is an Ivan-type theorem, see [5]. □

THEOREM 14. *Let $\varphi : [a,b] \to \mathbb{R}$ be a function and d a real number. If*
(1) *φ is continuous on $[a,b]$,*
(2) *$\varphi(a) \neq \varphi(b)$,*
(3) *$\varphi(x) \neq d$ for any $x \in [a,b]$,*
then there exists $\xi \in (a,b)$ such that

$$\frac{a\varphi(b)-b\varphi(a)}{\varphi(b)-\varphi(a)} - \xi + \frac{\varphi(\xi)}{\varphi'(\xi)} = d\left(\frac{1}{\varphi'(\xi)} - \frac{b-a}{\varphi(b)-\varphi(a)}\right). \tag{18}$$

Proof. In Theorem 13 we take $f : [a,b] \times [a',b'] \to \mathbb{R}$, $f(x,y) = \varphi(x)$. □

THEOREM 15. (Ivan). *Let $\varphi : [a,b] \to \mathbb{R}$ be a function satisfying*
(1) *φ is continuous on $[a,b]$,*
(2) *$\varphi(a) \neq \varphi(b)$,*
(3) *$\varphi(x) \neq 0$ for any $x \in [a,b]$.*
Then there exists $\xi \in (a,b)$ such that

$$\frac{a\varphi(b)-b\varphi(a)}{\varphi(b)-\varphi(a)} = \xi - \frac{\varphi(\xi)}{\varphi'(\xi)}. \tag{19}$$

Proof. In Theorem 14 let $d = 0$. □

REMARK 8. Other proof for Theorem 8 and Theorem 14 may be found in [9]. □

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