ERRORS ESTIMATION FOR IMPLICIT MANN ITERATION∗

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Abstract. We study the errors estimation for the implicit Mann iteration.


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Let $X$ be a real Banach space and $T : X \to X$ a map. Let $x_0 = x_0 \in X$. The following iteration is known as Mann iteration, see [1]:

$$y_{n+1} = (1 - \alpha_n)y_n + \alpha_n Ty_n.$$  (1)

For the rest of the paper, we suppose that there exists $(I - tT)^{-1}$, for all $t \in [0,1[$. The implicit Mann iteration was considered for the first time in [2]:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_{n+1}.$$  (2)

The sequence $\{\alpha_n\}$ is in $(0,1)$.

Remark 1. In order to have a well defined sequence $\{x_n\}$, the existence of $(I - \lambda T)^{-1}$, $\forall \lambda \in ]0,1[$, is crucial. Take $X = \mathbb{R}$, $Tx = x^4$, $\alpha_0 = 1/3$, $x_0 = 4$, or $X = \mathbb{R}$, $Tx = x^4$, $\alpha_0 = 1/2$, $x_0 = 2$, to see that there are no real values for $x_1$ that satisfy (2), that is $x_1 = (1 - \alpha_0)x_0 + \alpha_0 x_1^4$.

Estimations for the errors of Mann iteration are given in [3]. We aim to study the errors for the implicit iteration (2). Consider

$$x_{n+1} = (1 - \alpha_n) F_n x_n,$$

where $F_n = (I - \alpha_n T)^{-1}$, $\forall n \in \mathbb{N}$. Define the errors of $F_n x_n$ by

$$u_n = F_n x_n - F_n x_n,$$

where $\overline{F_n x_n}$ is the exact value of $F_n x_n$, that is $F_n x_n$ is an approximation value of $\overline{F_n x_n}$. The theory of errors shows that $\{u_n\}$ is bounded. Set

$$M = \sup \{||u_n|| : n \in \mathbb{N}\}.$$  

The errors from (2) come essentially from $F_n x_n$. This errors will affect the next step $n + 1$.

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Theorem 1. Let $X$ be a real Banach space and $T : X \to X$ a map. Let $x_0 = x_0 \in X$. Suppose that there exists $(I - tT)^{-1}$, for all $t \in [0,1[$, such that (2) is well defined. Then the errors $E_n = x_{n+1} - x_{n+1}$ satisfy

\[ \|E_n\| \leq (1 - \alpha_n) M, \forall n \in \mathbb{N}. \]

Proof. We have

\[ x_{n+1} = (1 - \alpha_n) F_n x_n = (1 - \alpha_n) \overline{F_n x_n} + (1 - \alpha_n) u_n \]
\[ = x_{n+1} + (1 - \alpha_n) u_n. \]

Note that the computational errors exist at step $n$ by computing the $\overline{F_n x_n}$, hence it is inappropriate to consider

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_{n+1} + u_n. \]

Such trend could occur by analogy with Mann iteration with errors,

\[ y_{n+1} = (1 - \alpha_n)y_n + \alpha_n T y_n + u_n. \]

In (1) it makes sense to add the errors, but for implicit Mann iteration (2) it is not the case to consider (4).

References


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