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ERRORS ESTIMATION FOR IMPLICIT MANN ITERATION*

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Abstract. We study the errors estimation for the implicit Mann iteration.

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Let X be a real Banach space and $T : X \to X$ a map. Let $\overline{x_0} = x_0 \in X$. The following iteration is known as Mann iteration, see [1]:

$$y_{n+1} = (1 - \alpha_n)y_n + \alpha_n T y_n. \tag{1}$$

For the rest of the paper, we suppose that there exists $(I - tT)^{-1}$, for all $t \in]0, 1[$. The implicit Mann iteration was considered for the first time in [2]:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_{n+1}. \tag{2}$$

The sequence $\{\alpha_n\}$ is in $(0, 1)$.

Remark 1. *In order to have a well defined sequence* $\{x_n\}$, *the existence of* $(I - \lambda T)^{-1}, \forall\lambda \in]0, 1[$, *is crucial. Take* $X = \mathbb{R}$, $Tx = x^4$, $\alpha_0 = 1/3$, $x_0 = 4$, *or* $X = \mathbb{R}$, $Tx = x^4$, $\alpha_0 = 1/2$, $x_0 = 2$, *to see that there are no real values for* x_1 *that satisfy* (2), *that is* $x_1 = (1 - \alpha_0)\, x_0 + \alpha_0 x_1^4$.

Estimations for the errors of Mann iteration are given in [3]. We aim to study the errors for the implicit iteration (2). Consider

$$x_{n+1} = (1 - \alpha_n)\, F_n x_n,$$

where $F_n = (I - \alpha_n T)^{-1}, \forall n \in N$. Define the errors of $F_n x_n$ by

$$u_n = F_n x_n - \overline{F_n x_n},$$

where $\overline{F_n x_n}$ is the exact value of $F_n x_n$, that is $F_n x_n$ is an approximation value of $\overline{F_n x_n}$. The theory of errors shows that $\{u_n\}$ is bounded. Set

$$M = \sup\{\|u_n\| : n \in \mathbb{N}\}.$$

The errors from (2) come essentially from $F_n x_n$. This errors will affect the next step $n + 1$.

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Theorem 1. *Let X be a real Banach space and $T : X \to X$ a map. Let $\overline{x_0} = x_0 \in X$. Suppose that there exists $(I - tT)^{-1}$, for all $t \in]0,1[$, such that (2) is well defined. Then the errors $E_n = x_{n+1} - \overline{x_{n+1}}$ satisfy*

$$\|E_n\| \leq (1-\alpha_n)\, M, \forall n \in \mathbb{N}. \tag{3}$$

Proof. We have

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, F_n x_n = (1-\alpha_n)\, \overline{F_n x_n} + (1-\alpha_n)\, u_n \\ &= \overline{x_{n+1}} + (1-\alpha_n)\, u_n. \end{aligned}$$

□

Note that the computational errors exist at step n by computing the $\overline{F_n x_n}$, hence it is unappropriate to consider

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_{n+1} + u_n. \tag{4}$$

Such trend could occur by analogy with Mann iteration with errors,

$$y_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n + u_n.$$

In (1) it makes sense to add the errors, but for implicit Mann iteration (2) it is not the case to consider (4) .

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