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REFINEMENT OF SOME INEQUALITIES FOR MEANS

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Abstract. We consider weighted arithmetic means as, for example, $\alpha G+(1-\alpha)C$, with $\alpha\in(0,1)$, G, C being the geometric and anti-harmonic means, and we find the range of values of α for which the weighted mean is still greater or less than some suitable means, in this case the arithmetic and Hölder ones.

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1. INTRODUCTION

It is well-known that the classical means, namely the arithmetic, geometric and harmonic ones,

$$A=\frac{a+b}{2},\quad G=\sqrt{ab},\quad H=\frac{2ab}{a+b}$$

satisfy the inequalities

$$H<G<A,$$

for $0<a<b$. As in [2], we shall consider some other means, like:

– the Hölder and the anti-harmonic mean

$$Q=\left(\frac{a^2+b^2}{2}\right)^{1/2},\quad C=\frac{a^2+b^2}{a+b};$$

– the Pólya & Szegő logarithmic mean, the exponential (or identric), and the weighted geometric mean

$$L=\frac{b-a}{\ln b-\ln a},\quad I=\frac{1}{\mathrm{e}}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},\quad S=\left(a^ab^b\right)^{1/(a+b)}.$$

An exhaustive bibliography and a full treatment of the topic can be found in [1].

In [2], beside the known inequalities

$$H<G<L<I<A<Q<S<C, \tag{1}$$

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the authors present the following relation between some means

$$\frac{G+Q}{2} < A < \frac{G+C}{2} < Q < \frac{A+C}{2} < S. \tag{2}$$

We can add to (2) the following inequalities

$$L < \frac{G+A}{2} < \frac{G+Q}{2}. \tag{3}$$

The first one will follow from Proposition 1 of the next section (the second is obvious in view of (1)).

In section 2 we shall consider weighted arithmetic means as, for example, $\alpha G + (1-\alpha)C$, with $\alpha \in (0,1)$ instead of $(G+C)/2$, and we shall find the range of values of α for which the weighted mean is still greater or less than its neighbours in (2) or (3).

In section 3 we prove that $\alpha A + (1-\alpha)S > Q$ if and only if $\alpha \geq 2-\sqrt{2}$. As a consequence, there are numbers $0 < a < b$ for which $(A+S)/2 > Q$, while for other pairs of numbers $(A+S)/2 < Q$.

2. REFINED INEQUALITIES

Let us denote $t = b/a$, $t > 1$. It it obvious that, if $M(a,b)$ is any mean, it suffices to prove the inequalities in (1), (2) or (3) for $M(1,t)$. We shall write from now on $M(t)$ instead of $M(1,t)$.

PROPOSITION 1. 1. *$L(t) < \alpha G(t) + (1-\alpha)A(t)$, $\forall t > 1$ if and only if $\alpha \leq \frac{2}{3}$;*
2. $\alpha G(t) + (1-\alpha)Q(t) < A(t)$, $\forall t > 1$ if and only if $\alpha \geq \frac{1}{2}$.

Proof. 1. We have $L(t) < \alpha G(t) + (1-\alpha)A(t)$ if and only if

$$\frac{A(t)-L(t)}{A(t)-G(t)} > \alpha.$$

We denote

$$f_{11}(t) = \frac{A(t)-L(t)}{A(t)-G(t)} = \frac{(t+1)\ln t - 2(t-1)}{(t+1-2\sqrt{t})\ln t}. \tag{4}$$

The limits at 1 and ∞ are $\lim_{t\to 1} f_{11}(t) = 2/3$ and $\lim_{t\to\infty} f_{11}(t) = 1$. We evaluate $f_{11}(t) - 2/3$ and show that it is positive. The denominator is obviously positive; we substitute $u = \sqrt{t}$ in the numerator and obtain

$$f(u) = (u^2 + 4u + 1)\ln u - 3u^2 + 3.$$

We have $f(1) = f'(1) = f''(1) = 0$ and $f'''(u) = 2(u-1)^2/u^3 > 0$ for $u > 1$, hence $f_{11}(t) > 2/3$ for $t > 1$.

2. Let us consider for $t > 1$, the function

$$f_{12}(t) = \frac{Q(t)-A(t)}{Q(t)-G(t)} = \frac{\sqrt{2(t^2+1)}-(t+1)}{\sqrt{2(t^2+1)}-2\sqrt{t}} = 1 - \frac{\sqrt{t^2+1}+\sqrt{2t}}{\sqrt{2}(\sqrt{t}+1)^2}. \tag{5}$$

We have $f_{12}(t) < 1/2$, since $\sqrt{t^2+1} + \sqrt{2t} > \sqrt{2}/2\ (\sqrt{t}+1)^2 \Leftrightarrow \sqrt{t^2+1} > \sqrt{2}/2\ (t+1) \Leftrightarrow (t-1)^2 > 0$, and the conclusion follows since $\lim_{t\to 1} f_{12}(t) = 1/2$. □

PROPOSITION 2. 1. *$A(t) < \alpha G(t) + (1-\alpha)C(t)$, $\forall t > 1$ if and only if $\alpha \le \frac{1}{2}$;*
2. *$\alpha G(t) + (1-\alpha)C(t) < Q(t)$, $\forall t > 1$ if and only if $\alpha \ge \alpha_0$, where $\alpha_0 = f_{22}(\sqrt{u_0}) = 0.3471574308...$, with u_0 the unique root of (8) which is greater than 1, and f_{22} defined in (7).*

Proof. 1. For $t > 1$ we define

$$f_{21}(t) = \frac{C(t) - A(t)}{C(t) - G(t)} = \frac{(t-1)^2}{2(t^2+1-\sqrt{t}(t+1))}. \tag{6}$$

It follows that $f_{21}(t) > 1/2$, since

$$f_{21}(t) - \frac{1}{2} = \frac{\sqrt{t}(\sqrt{t}-1)^2}{2(t^2+1-\sqrt{t}(t+1))} = \frac{\sqrt{t}}{2(t+\sqrt{t}+1)} > 0.$$

The infimum of f_{21} on $(1,\infty)$ is precisely $1/2$, because $\lim_{t\to 1} f_{21}(t) = 1/2$.

2. We consider now

$$f_{22}(t) = \frac{C(t) - Q(t)}{C(t) - G(t)} = \frac{1}{2}\frac{2(t^2+1) - (t+1)\sqrt{2(t^2+1)}}{t^2+1-(t+1)\sqrt{t}} \tag{7}$$

and obtain $\lim_{t\to 1} f_{22}(t) = 1/3$ and $\lim_{t\to\infty} f_{22}(t) = (2-\sqrt{2})/2$. In order to find the maximum of f_{22} we calculate the roots of the derivative of f_{22}. Denoting by $u = \sqrt{t}$, we obtain a unique root in $(1,\infty)$ of f'_{22} from

$$u^8 - 8u^5 - 10u^4 - 8u^3 + 1 = 0, \tag{8}$$

which is $u_0 = 2.3859965175...$, for which $f_{22}(\sqrt{u_0}) = 0.3471574308\ldots$. The function f_{22} will increase up to $f_{22}(\sqrt{u_0}) = \alpha_0$ and then will decrease to $(2-\sqrt{2})/2$. □

LEMMA 3. *For $t > 1$, the following inequality holds*

$$t^{\frac{t}{t+1}} > t - \ln t. \tag{9}$$

Proof. The inequality (9) is equivalent to

$$\frac{t}{t+1}\ln t > \ln(t - \ln t).$$

We consider the function

$$k(t) = \ln(t-\ln t) - \frac{t-1}{t}\ln t, \quad t > 1,$$

with

$$k'(t) = \frac{(\ln t - 1)\ln t}{t^2(t - \ln t)}.$$

It has $\lim_{t\to 1} k(t) = 0$, $\lim_{t\to\infty} k(t) = 0$ and a minimum at $t_0 = \mathrm{e}$. It follows that $k(t) < 0$ on $(1,\infty)$, hence $((t-1)/t)\ln t > \ln(t - \ln t)$. It follows that

$$\frac{t}{t+1}\ln t > \frac{t-1}{t}\ln t > \ln(t-\ln t). \qquad \square$$

Now we can prove

Proposition 4. 1. *$Q(t) < \alpha A(t) + (1-\alpha)C(t)$, $\forall t > 1$ if and only if $\alpha \le \frac{1}{2}$;*
2. $\alpha A(t) + (1-\alpha)C(t) < S(t)$, $\forall t > 1$ if and only if $\alpha \ge \frac{1}{2}$.

Proof. 1. Let us consider, for $t > 1$

$$f_{41}(t) = \frac{C(t) - Q(t)}{C(t) - A(t)} = \frac{2(t^2+1) - (t+1)\sqrt{2(t^2+1)}}{(t-1)^2}. \tag{10}$$

It follows that

$$f_{41}(t) - \frac{1}{2} = \frac{3(t^2+1) + 2t - 2(t+1)\sqrt{2(t^2+1)}}{2(t-1)^2} \ge 0,$$

because $(3(t^2+1)+2t)^2 - 8(t+1)^2(t^2+1) \ge 0 \Leftrightarrow (t-1)^4$. We have $\lim_{t\to 1} f_{41}(t) = 1/2$, hence this is the infimum of f_{41} on $(1,\infty)$.

2. Finally we define

$$f_{42}(t) = \frac{C(t) - S(t)}{C(t) - A(t)} = 2\frac{t^2 + 1 - (t+1)t^{\frac{t}{t+1}}}{(t-1)^2}. \tag{11}$$

We have

$$f_{42}(t) - \frac{1}{2} = \frac{3(t^2+1) + 2t - 4(t+1)t^{\frac{t}{t+1}}}{4(t-1)^2}.$$

We consider the function

$$g(t) = \frac{3(t^2+1) + 2t}{4(t+1)} - t^{\frac{t}{t+1}}$$

and the numerator of its derivative

$$g_1(t) = 4t^{\frac{t}{t+1}}(t + 1 + \ln t) + 1 - 3t^2 - 6t. \tag{12}$$

Using the fact that $S > Q$, i.e., $t^{t/(t+1)} > \sqrt{(t^2+1)/2}$, we obtain that $g_1(t) > \sqrt{2(t^2+1)}g_2(t)$, where $g_2(t) = 2(t+1+\ln t) - (3t^2+6t-1)/\sqrt{2(t^2+1)}$. The numerator of g_2' is $(t+1)\big(\sqrt{2(t^2+1)}\big)^3 - (3t^4 + 7t^2 + 6t)$ and it is positive on $(1,10)$. It follows that $g_2(t) > g_2(1) = 0$, therefore g_1 is positive for $1 < t < 10$.

Let us consider now that $t \ge 10$. Using (9) in (12) we obtain that $g_1(t) > g_3(t) = t^2 - 2t + 2 - (2\ln t + 1)^2$. For $g_4(t) = \sqrt{t^2 - 2t + 2} - 2\ln t - 1$, the sign of g_4' is given by $t^2 - t - 2\sqrt{t^2 - 2t + 2}$; but $(t^2 - t)^2 - 4(t^2 - 2t + 2) = (t-10)^4 + 38(t-10)^3 + 537(t-10)^2 + 3348(t-10) + 7772 > 0$ for $t \ge 10$. It follows that $g_3(t) \ge g_3(10) = 3.45... > 0$, hence g_1 is positive for $t \ge 10$ too.

In conclusion, $g(t) > g(1) = 0$, and $f_{42}(t) > 1/2$ for $t > 1$; in addition, $\lim_{t\to 1} f_{42}(t) = 1/2$. $\square$

3. ANOTHER INEQUALITY

In general it is not an easy task to find the range where the parameter α may vary. To this aim the Symbolic Algebra Programs as Maple can be of great help.

We consider the following problem:

Find the values of $\alpha \in (0,1)$ for which

$$Q(t) - \alpha A(t) - (1-\alpha)S(t) > 0, \quad \forall t > 1. \tag{13}$$

In order to find the minimal value of α for which (13) holds, we shall develop asymptotically the function $F(t,\alpha) = Q(t) - \alpha A(t) - (1-\alpha)S(t)$ for $t \to \infty$. We denote

$$u = F(t,\alpha) = \frac{1}{2}\sqrt{2+2\,t^2} - \frac{1}{2}\alpha\,(1+t) - (1-\alpha)\,t^{\frac{t}{1+t}}. \tag{14}$$

Using the command `asympt(u,t,3)` we obtain the series

$$\begin{aligned}&\left(\frac{\sqrt{2}}{2} + \frac{\alpha}{2} - 1\right)t - \frac{\alpha}{2} + (1-\alpha)\ln(t)\\ &+\frac{\sqrt{2}/4 - (1-\alpha)\left(\ln(t) + 1/2\,(\ln(t))^2\right)}{t} + \mathcal{O}\left(t^{-2}\right).\end{aligned} \tag{15}$$

(We mention that the term $\mathcal{O}\left(t^{-2}\right)$ is used in the sense of Maple; from the point of view of Landau notation it should be $\mathcal{O}(t^{-2}\ln^3(t))$.)

For $u > 0$, the condition $\sqrt{2}/2 + \alpha/2 - 1 \geq 0$, hence $\alpha \geq 2 - \sqrt{2}$, is obviously necessary. Now that we know the expected minimal value of α, we state

THEOREM 5. *The inequality* (13) *holds if and only if* $\alpha \geq 2 - \sqrt{2}$.

Proof. We shall prove that the inequality holds for $\alpha = 2 - \sqrt{2}$ (hence *a fortiori* for $\alpha \geq 2 - \sqrt{2}$).

Let us denote $f(t) = (\sqrt{2}+1)F(t, 2-\sqrt{2})$, where F is given in (14). We have to prove that $f(t) > 0$ for $t > 1$. It follows that

$$f(t) = \frac{(\sqrt{2}+1)\sqrt{2(t^2+1)}}{2} - \frac{\sqrt{2}(t+1)}{2} - t^{\frac{t}{t+1}}.$$

We put in the inequality $(1+x)^q < 1 + qx$, which holds for $x > 0$, $0 < q < 1$, $x = t - 1$ and $q = t/(t+1)$. It follows that

$$t^{\frac{t}{t+1}} < \frac{t^2+1}{t+1},$$

and

$$f(t) > \frac{(1+\sqrt{2})}{2(t+1)}\left((t+1)\sqrt{2+2t^2} - \sqrt{2}(t^2 + 2(\sqrt{2}-1)t + 1)\right).$$

Let us denote the positive expressions

$$f_1(t) = (t+1)\sqrt{2+2t^2}, \quad f_2(t) = \sqrt{2}(t^2 + 2(\sqrt{2}-1)t + 1);$$

it follows easily that $f_1^2(t) - f_2^2(t) = 4t(t-1)^2$, therefore $f(t) > 0$. □

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